Lecture 6.1, MATH-57091 Probability and Statistics for High-School Teachers.

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Two Very Cool inequalities with very Russian names.

Markov's Inequality

If X is a random variable that takes only nonnegative values, the for any value $a > 0$:

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next we notice that we integrate over the interval $[a, \infty)$ so on this interval $x \geq a$:

$$\geq \int_{a}^{\infty} af(x)dx = a\int_{a}^{\infty} f(x)dx = aP(X \geq a).$$

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Example

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Hence,

$$P(|X - 500| \leq 100) = 1 - P(|X - 500| \geq 100) \geq 1 - \frac{1}{100} = .99.$$