Analysis Qualifying Exam, 17th January 2015.

1. For every positive $r \neq 2$, evaluate

$$\int_\gamma \frac{z^2+1}{z(z^2+4)}dz$$

where $\gamma(t) = re^{it}, 0 \leq t \leq 2\pi$.

2. Let $\lambda > 1$. Show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z : \text{Re}z > 0\}$, and that this solution must be real.

3. Let f be analytic in $G = \{z : 0 < |z-a| < 1\}$ except that there is a sequence of isolated essential singularities a_n in G with $a_n \to a$. Prove that for any complex number ω there is a sequence $\{z_n\}$ in G with $a = \lim_{n\to\infty} z_n$ and $\omega = \lim_{n\to\infty} f(z_n)$.

4. Suppose f is analytic in the closed unit disc $\{z : |z| \leq 1\}$ and $|f(z)| = 1$ when $|z| = 1$. Suppose that f has zero at $z = \frac{1}{4}(1+i)$ and a double zero at $z = \frac{1}{2}$. Can $f(0) = \frac{1}{8}$?

5. Map the exterior of the unit circle with a cut along the ray $[1, \infty)$ onto the upper half-plane.

6. (a) State the finite version of the Besicovitch covering lemma.

(b) State *and prove* the finite version of the Vitali covering lemma.

7. Suppose that, for each $j \in \mathbb{N}$, $f_j : [0,1] \to \mathbb{R}$ is a Lebesgue measurable function satisfying $0 \leq f_j \leq \frac{3}{2}$ and

$$\int_0^1 f_j \, dm = 1.$$

Prove that

$$m\left(\{x \in [0,1] : \limsup_{j\to\infty} f_j(x) \geq \frac{1}{2}\}\right) \geq \frac{1}{2}.$$

8. The domain Ω in $\mathbb{R}^3$ is given by

$$\{(x,y,z) : 0 \leq x < y,\ 2z \leq x^2 + y^2\}.$$

Find the limits one should use in the repeated integral

$$\int \left[\int \left(\int f(x,y,z)dx\right) dy\right] dz$$

so that it represents the Lebesgue integral of f over Ω.

9. The integral

$$\int_{[0,1]^2} f(x^3+y^2, x^2)\,dxdy$$

can be written as

$$\int_{\Omega} f(z,t)G(z,t)\,dzdt.$$

What are Ω and G in this formula?

10. Let $(f_j)_{j\in\mathbb{N}}$ be a sequence of Lebesgue measurable functions satisfying

$$\sup_{j\in\mathbb{N}} \int_1^\infty f_j^2(x)\,dx \leq 1$$

such that f_j converges pointwise to a function f. Prove that

$$\lim_{j\to\infty} \int_1^\infty \frac{f_j(x)}{x}\,dx = \int_1^\infty \frac{f(x)}{x}\,dx.$$