

## Analysis Qualifying Exam. January 2017.

1. Let  $E \subset \mathbf{R}$  be a set of measure zero. Show that there exists a function defined on  $\mathbf{R}$ , which is continuous and increasing everywhere and that fails to be differentiable at each point in  $E$ .

2. Let  $(X, \mathcal{M}, \mu)$  be a measure space such that  $\mu(X) = 1$ . For each  $1 < p < \infty$  define the set

$$E_p = \{f \in L^1(\mu) : \int |f| d\mu = 1 \text{ and } \int |f|^p d\mu = 2\}.$$

Show that for each  $0 < \varepsilon < 1$  there exists some  $\delta_{p,\varepsilon} > 0$  such that

$$\mu(\{x \in X : |f(x)| > \varepsilon\}) \geq \delta_{p,\varepsilon}$$

for each  $f \in E_p$ .

3. Show that the set of irrational numbers is not a countable union of closed subsets of  $\mathbf{R}$ .

4. Show that if any continuous real-valued function on a metric space  $X$  attains its maximum value, then  $X$  is totally bounded.

5. Let  $\{\phi_n\}$  be an orthonormal sequence in a Hilbert space  $H$  and consider the operator  $T : H \rightarrow H$  defined by

$$T(f) = \sum_{n=1}^{\infty} \alpha_n \phi_n,$$

where  $\{\alpha_n\}$  is a sequence of scalars satisfying  $\lim \alpha_n = 0$ . Show that  $T$  is a compact operator.

6. Suppose that  $f$  is holomorphic in an open neighbourhood of the closure of the disc  $D(z_0, r)$ , with  $z_0 \in \mathbb{C}$ ,  $r > 0$ . Further assume that  $f(z) \neq 0$  whenever  $|z - z_0| = r$ . Show that

$$\int_{\{|z-z_0|=r\}} \frac{f'(z)}{f(z)} dz = 2\pi Ni$$

where  $N$  is the number of zeroes of  $f$  inside  $D(z_0, r)$ . The contour integral around the circle  $\{|z - z_0| = r\}$  is taken counter-clockwise.

7. Evaluate the limit

$$\lim_{N \rightarrow \infty} \int_{1/N}^N \sin(x^2) dx,$$

carefully justifying your answer. You may use the fact that  $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$  without proof.

8. Find a conformal mapping from the half plane with a cut given by

$$\{z = x + iy : x > 0\} \setminus [0, 1]$$

onto the upper half disc  $\{z = x + iy : y > 0, \text{ and } x^2 + y^2 < 1\}$ .

9. Suppose that  $f$  is entire and satisfies

$$f(z + i) = e^{2\pi} f(z) \text{ and } f(z + 1) = f(z)$$

for every  $z \in \mathbb{C}$ . Prove that

$$f(z) = \alpha e^{-2\pi iz}$$

for some  $\alpha \in \mathbb{C}$ .

10. Suppose that  $f_n$  is a sequence of entire functions that converge uniformly on compact sets to a function  $f$ . Suppose that  $f_n(z) \neq 0$  for every  $n \in \mathbb{N}$  and  $z \in \mathbb{D}$  (the unit disc), but  $f(0) = 0$ . Prove that  $f(z) = 0$  for every  $z \in \mathbb{C}$ .