Differential Geometry, MATH-45011/55011.
Home Work 1, due on Wednesday, SEPTEMBER 4
Instructor: Prof. Artem Zvavitch

Problem 1. *Find parametrization of the following curves*

- $y^2 - x^2 = 1.$
- $x^3 + y^3 = 3xy.$
- $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$, *where* $p > q > 0.$

Problem 2. *Consider the ellipse* $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$, *where* $p > q > 0$. *The points* $\mathbf{f_1} = (-\sqrt{p^2 - q^2}, 0)$ *and* $\mathbf{f_2} = (-\sqrt{p^2 - q^2}, 0)$ *are called foci of the ellipse. Prove that the sum of the distances from* $\mathbf{f_1}$ *and* $\mathbf{f_2}$ *to any point* α *on the ellipse does not depend on* α.

Problem 3. *This exercise shows that a straight line is the shortest curve joining two given points. Let* P *and* Q *be the two points, and let* $\alpha(t) : [a, b] \to \mathbb{R}^3$ *be a curve such that* $\alpha(a) = P$ *and* $\alpha(b) = Q$.

- *Show that for any unit vector* u*:* $\alpha'(t) \cdot u \leq |\alpha'(t)|$ *(Hint: use Cauchy-Schwarz inequality).*
- *Use inequality* $|\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$ *and the above statement to show that*
$$(Q - P) \cdot u \leq \int_a^b |\alpha'(t)|dt.$$
- *Finally take* $u = (Q - P)/|Q - P|$ *to finish the proof.*

Problem 4. *Find the arc-length of the curve* $\alpha(t) = (3t^2, t - 3t^3)$, *starting at* $t = 0$.

Problem 5. *Consider a curve* $\alpha(t) = (cos^2 t, sin^2 t)$ *for* $t \in (0, \pi/2)$. *Check if this curve is regular or not, if yes find the unit-speed reparametrization of this curve.*

Problem 6. *Show that the volume* V *of a parallelepiped generated by three linearly independent vectors* $u, v, w \in \mathbb{R}^3$ *is given by*
$$V = |(u \wedge v) \cdot w|.$$