Differential Geometry, MATH-45011/55011.
Home Work 4, due on Wednesday, SEPTEMBER 25
Instructor: Prof. Artem Zvavitch

Problem 1. *Let $\boldsymbol{\alpha}(t)$ be a regular curve. Suppose that there is a point $\boldsymbol{a} \in \mathbb{R}^3$ such that $\boldsymbol{\alpha}(t) - \boldsymbol{a}$ is orthogonal to $\boldsymbol{T}(t)$ for all t. Prove that $\boldsymbol{\alpha}(t)$ lies on the sphere.*

Problem 2. *Let $\boldsymbol{\alpha}(t)$ be a regular curve and let $\boldsymbol{a}$ be a point that belongs to each normal plane of $\boldsymbol{\alpha}$. Prove that $\boldsymbol{\alpha}$ lies on the sphere.*

Problem 3. *(***This is EXTRA PROBLEM +5 points for exam for each question =10 Extra points***)*
a) Prove that if all tangent lines of a given curve are parallel to a plane, then the curve is plane.
b) Prove that if all osculating planes of a given curve pass through a fixed point, then the curve is plane.

Problem 4. *Let $\boldsymbol{\alpha}(s)$ be a unit speed curve with $(\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B})$ and $k > 0$ and $\tau > 0$. Define $\boldsymbol{\beta}(s) = \int_0^s \boldsymbol{B}(\sigma)d\sigma$. Prove that $\boldsymbol{\beta}$ is unit speed and show that the Frenet-Serret apparatus $\{\bar{k}, \bar{\tau}, \bar{\boldsymbol{T}}, \bar{\boldsymbol{N}}, \bar{\boldsymbol{B}}\}$ of $\boldsymbol{\beta}$ satisfies $\bar{k} = \tau$, $\bar{\tau} = k$, $\bar{\boldsymbol{T}} = \boldsymbol{B}$, $\bar{\boldsymbol{N}} = -\boldsymbol{N}$, and $\bar{\boldsymbol{B}} = \boldsymbol{T}$.*

Problem 5. *a) Let $S^2 = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\}$ and $\mathbb{R}^2 = \{(u, v, w) \in \mathbb{R}^3 : w = 0\}$. If $(u, v, 0)$ belongs to $\mathbb{R}^2$, the line determined by $(u, v, 0)$ and $(0, 0, 1)$ intersects S^2 in a point other than $(0, 0, 1)$. Denote this point by $\boldsymbol{r}(u, v)$. Compute the actual form of $\boldsymbol{r}(u, v)$ and show that $\boldsymbol{r} : \mathbb{R}^2 \to \mathbb{R}^3$ is a simple surface. The inverse mapping to $\boldsymbol{r}$ is called the stereographic projection.*
b) Let $\boldsymbol{\alpha} : (a, b) \to \mathbb{R}^3$ be a unit speed curve with $k \neq 0$ and let

$$\mathcal{U} = \{(u, v) \in \mathbb{R}^2 : a < u < b,\ v \neq 0\}.$$

Define $\boldsymbol{r} : \mathcal{U} \to \mathbb{R}^3$ by $\boldsymbol{r}(u, v) = \boldsymbol{\alpha}(u) + v\boldsymbol{\alpha}'(u)$. Prove that $\boldsymbol{r}$ is a simple surface, provided $\boldsymbol{r}$ one-to-one. It is called the tangent developable surface of $\boldsymbol{\alpha}$.
c) Let

$$\boldsymbol{r}(u, v) = (\sin u \cos v, 2 \sin u \sin v, 3 \cos u), \qquad -1 < u < 1,\ 0 < v < \pi.$$

Show that $\boldsymbol{r}$ is a simple surface. What is it?