

Analysis (42001/52001)

Refresh for Exam 1

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Do it and get 20 bonus points Due before the exam on October 8, BUT better do it before October 1 and ask Artem questions!

**Problem 1.** Prove, using the method of mathematical induction,

$$\sum_{k=1}^n (2k-1)^2 = \frac{n(4n^2-1)}{3}.$$

**Problem 2.** Prove that the set of all finite sequences of English letters is countable.

**Note:** One may also say “finite words” instead of “finite sequences,” provided we accept that any combination of English letters (not just dictionary words) may be regarded as a word in this mathematical sense. :-)

**Problem 3.** (1) Prove that for any positive numbers  $a, b, c, d$  if  $a < b$  and  $c < d$ , then  $ac < bd$ .

(2) Prove that  $1.4 < \sqrt{2} < 1.5$ .

**Problem 4.** Consider a set  $S$  of all rational numbers in  $(0, 1)$  i.e.

$$S = \{x \in (0, 1) : x \text{ is a rational number}\}$$

or

$$S = (0, 1) \cap \mathbb{Q}.$$

Find  $\sup S$ . **DO NOT FORGET TO PROVE THAT THE NUMBER YOU PROPOSE IS THE  $\sup S$ .**

**Problem 5.** Proof that

$$\sup_D |f(x) + g(x)| \leq \sup_D |f(x)| + \sup_D |g(x)|.$$

Show an example of two functions  $f, g$  and a domain  $D$ , for which the above inequality is

(1) Equality.

(2) Strict inequality.

**Problem 6.** (Use the definition of convergent sequence to prove that  $a_n = \frac{n+(-1)^n}{n+2}$  converges to 1.

**Problem 7.** Use the definition of convergent sequence to prove that  $a_n = 3 + (-1)^n$  is divergent (i.e. not convergent).

**Problem 8.** Let's have fun and compute  $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}$ . More precisely, Define a sequence by

$$a_1 = 0, \quad a_{n+1} = \sqrt{2 + a_n} \quad \text{for } n \geq 1.$$

(1) Prove that  $(a_n)$  is monotone.

(2) Prove that  $(a_n)$  is bounded.

(3) Conclude that  $(a_n)$  converges and find  $\lim_{n \rightarrow \infty} a_n$ .

**Problem 9.** Is it true that any sequence contains convergent subsequence?