Functions of Real Variables II (62052/72052)
Final HW, DUE MAY 5.
Instructor: Prof. Artem Zvavitch.

Problem 1. *A normed space* $(V, \| * \|)$ *is separable if there exists a countable set of elements* $\{v_k\}_{k=1}^{\infty}$ *of* V *that is dense.*

A measure space (X, μ) *is* **separable** *if there is a countable family of measurable subsets* $\{E^k\}_{k=1}^{\infty}$ *so that if* E *is any measurable set of finite measure, then*

$$\mu(E \Delta E_{n_k}) \to 0 \text{ as } k \to 0,$$

for some subsequence of $\{n_k\}$*, which may depend on* E*.*

- *Show that* $(\mathbb{R}^d, \| * \|)$ *is separable, where* $\|x\| = (\sum x_i^2)^{1/2}$*.*
- *Show that* $\mathbb{R}^d$ *with usual Lebesque measure is separable measure space.*
- *Prove that if measure space* (X, μ) *is separable, then* $L^p(X, \mu)$ *is separable for all* $p \in [1, \infty)$*.*
- *Now give an example of not separable normed space: show that the space* $L^{\infty}(\mathbb{R})$ *is not separable, you may do it by constructing for each* $a \in \mathbb{R}$ *a function* $f_a \in L^{\infty}$ *such that* $\|f_a - f_b\|_{L^{\infty}} \geq 1$*, if* $a \neq b$*.*
- *Do the same for the dual space of* $L^{\infty}(\mathbb{R})$*.*
- *Study the above examples to answer the question if it is true that* $(B^*)^* = B$ *for any Banach space* B*.*

Problem 2. *Suppose the measure space* (X, μ) *is separable. Let* $1 \leq p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. A sequence* $\{f_n\}$ *with* $f_n \in L^p(X)$ *is said to converge to* $f \in L^p(X)$ **weakly** *if*

$$\int f_n g d\mu \to \int f g d\mu \text{ for every } g \in L^q(X).$$

- *Prove that* $f_n \to f$ *in* L^p *then* f_n *converges to* f *weakly.*
- *Suppose that* $\sup_n \|f_n\|_p < \infty$*. Then, use the separability of* L_q *to verify that weak convergence can be checked on a dance subset of functions* $g \in L^q$*.*
- *Suppose* $p \in (1, \infty)$*. Show that if* $\sup_n \|f_n\|_{L^p} < \infty$*, then there exists a subsequence of* $\{f_n\}$ *which converges to a function* $f \in L_p$ *weakly.*
- *Consider a sequence of function* $f_n = \sin(2\pi n x)$ *in* $L^p([0,1])$*. Show that* $f_n \to 0$*weakly (Hint: you may check it for simple functions* g*).*
- *Let* $\chi(t)$ *be a characteristic function of interval* $[0,1]$*. Show that* $f_n(x) = n^{1/p}\chi(nx)$ *is in* $L_p(\mathbb{R})$*. Also show that* $f_n \to 0$ *weakly for* $p > 1$*, but not for* $p = 1$*.*
- *Now consider* $f_n(x) = 1 + \sin(2\pi n x)$ *in* $L^1([0,1])$*. Then* $f_n \to 1$ *weakly in* $L_1([0,1])$*,* $\|f_n\|_{L^1} = 1$*, but* $\|f_n - f\|_{L^1}$ *does not converge to zero.*