THE BUSEMANN-PETTY PROBLEM FOR ARBITRARY MEASURES.

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Abstract. The Busemann-Petty problem asks whether symmetric convex bodies in $\mathbb{R}^n$ with smaller $(n-1)$-dimensional volume of central hyperplane sections necessarily have smaller n-dimensional volume. The answer to this problem is affirmative for $n \le 4$ and negative for $n \ge 5$. In this paper we generalize the Busemann-Petty problem to essentially arbitrary measure in place of the volume. We also present applications of the latter result by proving several inequalities concerning the measure of sections of convex symmetric bodies in $\mathbb{R}^n$.

1. Introduction

Consider a non-negative, even function $f_n(x)$, which is locally integrable on $\mathbb{R}^n$. Let μ_n be the measure on $\mathbb{R}^n$ with density f_n.

For $\xi \in S^{n-1}$, let $\xi^\perp$ be the central hyperplane orthogonal to ξ. Consider a non-negative even function f_{n-1} on $\mathbb{R}^n$, which is locally integrable on every $\xi^\perp$. We define a measure μ_{n-1} on $\xi^\perp$, for each $\xi \in S^{n-1}$, so that for every bounded Borel set $B \subset \xi^\perp$,

$$\mu_{n-1}(B) = \int_B f_{n-1}(x)dx.$$

We say that a closed set K in $\mathbb{R}^n$ is a **star body** if every straight line passing through the origin crosses the boundary of K at exactly two points, the origin is an interior point of K and the boundary of K is continuous.

In this paper we study the following problem.

The Busemann-Petty problem for general measures (BPGM): *Fix $n \ge 2$. Given two convex origin-symmetric star bodies K and L in $\mathbb{R}^n$ such that*

$$\mu_{n-1}(K \cap \xi^\perp) \le \mu_{n-1}(L \cap \xi^\perp)$$

for every $\xi \in S^{n-1}$, does it follow that

$$\mu_n(K) \le \mu_n(L)?$$

Clearly, the BPGM problem is a triviality for $n = 2$ and $f_{n-1} > 0$, and the answer is "yes", moreover $K \subseteq L$. Also note that this problem is a generalization of the Busemann-Petty problem, posed in 1956 (see [BP])

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and asking the same question for Lebesgue measures: $\mu_n(K) = \mathrm{Vol}_n(K)$ and $\mu_{n-1}(K \cap \xi^\perp) = \mathrm{Vol}_{n-1}(K \cap \xi^\perp)$ i.e. $f_n(x) = f_{n-1}(x) = 1$.

The Minkowski theorem (see [Ga3]) shows that an origin-symmetric star body is uniquely determined by the volume of its hyperplane sections (the same is true for the case of general symmetric measure, see Corollary 1 below). In view of this fact it is quite surprising that the answer to the original Busemann Petty problem is negative for $n \geq 5$. Indeed, it is affirmative if $n \leq 4$ and negative if $n \geq 5$. The solution appeared as the result of a sequence of papers: [LR] $n \geq 12$, [Ba] $n \geq 10$, [Gi] and [Bo2] $n \geq 7$, [Pa] and [Ga1] $n \geq 5$, [Ga2] $n = 3$, [Zh2] and [GKS] $n = 4$ (we refer to [Zh2], [GKS], [K9] and [K10] for more historical details).

It was shown in [Z], that the answer to BPGM in the case of the standard Gaussian measure ($f_n(x) = f_{n-1} = e^{-|x|^2/2}$) is the same: affirmative if $n \leq 4$ and negative if $n \geq 5$.

Answers to the original and Gaussian Busemann-Petty problems suggest that the answer to the BPGM could be independent from the choice of measures and depend only on the dimension n.

In Corollary 2 below we confirm this conjecture by proving the following:

Let $f_n(x) = f_{n-1}(x)$ be equal even non-negative continuous functions:

- *the answer to the BPGM problem is affirmative for $n \leq 4$,*
- *if f_{n-1} is a positive twice differentiable function on $\mathbb{R}^n \setminus \{0\}$, then the answer to the BPGM problem is negative for $n \geq 5$.*

Actually, the above fact is a corollary of a pair of more general theorems. Those theorems use the Fourier transform in the sense of distributions to characterize those functions $f_n(x)$ and $f_{n-1}(x)$ for which the BPGM problem has affirmative (or negative) answer in a given dimension:

Theorem 1. (BPGM: affirmative case) *Let f_n and f_{n-1} be even continuous non-negative functions such that*

$$t\frac{f_n(tx)}{f_{n-1}(tx)} \tag{1}$$

is a nondecreasing function of t for any fixed $x \in S^{n-1}$. Suppose that a symmetric star body K in $\mathbb{R}^n$ has the property that

$$\|x\|_K^{-1}\frac{f_n\left(\frac{x}{\|x\|_K}\right)}{f_{n-1}\left(\frac{x}{\|x\|_K}\right)} \tag{2}$$

is a positive definite distribution on $\mathbb{R}^n$. Then for any symmetric star body L in $\mathbb{R}^n$ satisfying

$$\mu_{n-1}(K \cap \xi^\perp) \leq \mu_{n-1}(L \cap \xi^\perp), \ \ \forall \xi \in S^{n-1},$$

we have

$$\mu_n(K) \leq \mu_n(L).$$

Theorem 2. (BPGM: negative case) *Let f_n and f_{n-1} be even continuous positive functions on $\mathbb{R}^n \setminus \{0\}$, such that*

$$t\frac{f_n(tx)}{f_{n-1}(tx)}$$

is a nondecreasing function of t for any fixed $x \in S^{n-1}$. Also assume that $f_{n-1}(x) \in C^2(\mathbb{R}^n \setminus \{0\})$.

If L is an infinitely smooth, origin symmetric, convex body in $\mathbb{R}^n$ with positive curvature, and the function

$$\|x\|_L^{-1}\frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)} \tag{3}$$

is in $C^{n-2}(\mathbb{R}^n \setminus \{0\})$ and does not represent a positive definite distribution, then there exists a convex symmetric body D in $\mathbb{R}^n$ such that

$$\mu_{n-1}(D \cap \xi^\perp) \le \mu_{n-1}(L \cap \xi^\perp), \;\; \forall \xi \in S^{n-1},$$

but

$$\mu_n(D) > \mu_n(L).$$

Theorems 1 and 2 are generalizations of a theorem of Lutwak (see [Lu]) who provided a characterization of symmetric star bodies for which the original Busemann-Petty problem has an affirmative answer (see [Z] for the case of Gaussian measure). Let K and M be symmetric star bodies in $\mathbb{R}^n$. We say that K is the intersection body of M if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the central hyperplane section of L perpendicular to this direction. A more general class of *intersection bodies* is defined as the closure in the radial metric of the class of intersection bodies of star bodies (see [Ga3], Chapter 8).

Lutwak ([Lu], see also [Ga2] and [Zh1]) proved that if K is an intersection body then the answer to the original Busemann-Petty problem is affirmative for every L, and, on the other hand, if L is not an intersection body, then one can perturb it to construct a body D giving together with L a counterexample.

Lutwak's result is related to Theorems 1 and 2 via the following Fourier analytic characterization of intersection bodies found by Koldobsky [K4]: an origin symmetric star body K in $\mathbb{R}^n$ is an intersection body if and only if the function $\|\cdot\|_K^{-1}$ represents a positive definite distribution on $\mathbb{R}^n$.

We present the proof of Theorems 1 and 2 in Section 3. The proof is based on the Fourier transform of distributions, the Spherical Parseval's identity introduced by Koldobsky (see Lemma 3 in [K5] or Proposition 1 in Section 3) and an elementary functional inequality (see Lemma 1).

Another application of Theorem 1 is motivated by a question of what one has to know about the measure of central sections of the bodies K and L to make a conclusion about the relation between the volumes of K and L in every dimension. Results of such a type, involving derivatives or the Laplace transform of the parallel sections functions were proved in [K5], [K7], [K8], [K9], [K10], [RZ], [KYY].

Note that Theorem 1 allows us to consider a different situation where measures μ_n and μ_{n-1} used for bodies and their sections, respectively, are different. This leads to a number of interesting facts and gives examples of non-trivial densities $f_n(x)$ and $f_{n-1}(x)$ for which the BPGM problem has an affirmative answer in any dimension (see Section 4). Probably the most notable statement is (see Corollary 4):

For any $n \geq 2$ and any symmetric star bodies $K, L \subset \mathbb{R}^n$ such that

$$\int_{K\cap\xi^\perp} \sum_{i=1}^{n} |x_i|\, dx \leq \int_{L\cap\xi^\perp} \sum_{i=1}^{n} |x_i|\, dx$$

for every $\xi \in S^{n-1}$, we have

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L).$$

One of the advantages of the latter result (and other results of this type) is that now one can apply methods and inequalities from asymptotic convex geometry (see [MP]) to produce bounds on the volume of hyperplane sections of a convex body (see Section 4).

Finally iterating Theorem 1 with different densities f_n and f_{n-1} we present some new results for sections of codimension greater than 1 (see Theorem 4 and Corollary 3).

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2. Measure of sections of star bodies and the Fourier transform

Our main tool is the Fourier transform of distributions (see [GS], [GV], [K9] and [K11] for exact definitions and properties). We denote by $\mathcal{S}$ the space of rapidly decreasing infinitely differentiable functions (test functions) on $\mathbb{R}^n$ with values in $\mathbb{C}$. By $\mathcal{S}'$ we denote the space of distributions over $\mathcal{S}$. The Fourier transform of a distribution f is defined by $\langle \hat{f}, \hat{\phi}\rangle = (2\pi)^n \langle f, \phi\rangle$, for every test function ϕ. A distribution f is called *even homogeneous* of degree $p \in \mathbb{R}$ if

$$\langle f(x), \phi(x/t)\rangle = |t|^{n+p}\langle f(x), \phi(x)\rangle, \;\; \forall \phi \in \mathcal{S},\; t \in \mathbb{R}\setminus\{0\}.$$

The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$.

A distribution f is called *positive definite* if, for every nonnegative test function $\phi \in S$,

$$\langle f, \phi * \overline{\phi(-x)}\rangle \geq 0.$$

By Schwartz's generalization of Bochner's theorem, a distribution is positive definite if and only if its Fourier transform is a positive distribution (in the sense that $\langle \hat{f}, \phi \rangle \geq 0$, for every non-negative $\phi \in S$). Every positive distribution is a tempered measure, i.e. a Borel non-negative, locally finite measure γ on $\mathbb{R}^n$ such that, for some $\beta > 0$,

$$\int_{\mathbb{R}^n} (1+|x|)^{-\beta} d\gamma(x) < \infty,$$

where $|x|$ stands for the Euclidean norm (see [GV] p. 147).

The *spherical Radon transform* is a bounded linear operator on $C(S^{n-1})$ defined by

$$\mathcal{R}f(\xi) = \int_{S^{n-1}\cap\xi^\perp} f(x)dx, \;\; f \in C(S^{n-1}), \;\; \xi \in S^{n-1}.$$

Koldobsky ([K1], Lemma 4) proved that if $g(x)$ is an even homogeneous function of degree $-n+1$ on $\mathbb{R}^n \setminus \{0\}$, $n > 1$ so that $g\big|_{S^{n-1}} \in L_1(S^{n-1})$, then

$$\mathcal{R}g(\xi) = \frac{1}{\pi}\hat{g}(\xi), \qquad \forall \xi \in S^{n-1}. \tag{4}$$

For a star body $K \subset \mathbb{R}^n$ the Minkowski functional of K is given by

$$\|x\|_K = \min\{\alpha > 0 : x \in \alpha K\}, \;\; x \in \mathbb{R}^n.$$

Theorem 3. *Let K be a symmetric star body in $\mathbb{R}^n$, then*

$$\mu_{n-1}(K\cap\xi^\perp) = \frac{1}{\pi}\Big[|x|^{-n+1}\int_0^{|x|/\|x\|_K} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right)dt\Big]^\wedge(\xi).$$

Proof : If χ is the indicator function of the interval $[-1,1]$ then, passing to the polar coordinates in the hyperplane $\xi^\perp$ we get

$$\mu_{n-1}(K\cap\xi^\perp) = \int_{(x,\xi)=0} \chi(\|x\|_K) f_{n-1}(x)dx = \int_{S^{n-1}\cap\xi^\perp}\int_0^{\|\theta\|_K^{-1}} t^{n-2} f_{n-1}(t\theta)dtd\theta.$$

We extend the function under the spherical integral to a homogeneous of degree $-n+1$ function on $\mathbb{R}^n$ and apply (4) to get

$$\begin{aligned}\mu_{n-1}(K\cap\xi^\perp) &= \int\limits_{S^{n-1}\cap\xi^\perp} |x|^{-n+1}\int\limits_0^{|x|/\|x\|_K} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right)dtdx\\ &= \mathcal{R}\Big[|x|^{-n+1}\int\limits_0^{|x|/\|x\|_K} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right)dt\Big](\xi)\\ &= \frac{1}{\pi}\Big[|x|^{-n+1}\int\limits_0^{|x|/\|x\|_K} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right)dt\Big]^\wedge(\xi). \qquad (5)\end{aligned}$$

□

Theorem 3 implies that a symmetric star body is uniquely determined by the measure μ_{n-1} of its sections:

Corollary 1. *Assume $f_{n-1}(x)\neq 0$ everywhere except for a countable set of points in $\mathbb{R}^n$. Let K and L be origin symmetric star bodies in $\mathbb{R}^n$. If*

$$\mu_{n-1}(K\cap\xi^\perp)=\mu_{n-1}(L\cap\xi^\perp),\quad \forall\xi\in S^{n-1},$$

then $K=L$.

Proof : Note that the function in (5) is homogeneous of degree -1 (with respect to $\xi\in\mathbb{R}^n$). This gives a natural extension of $\mu_{n-1}(K\cap\xi^\perp)$ to a homogeneous function of degree -1. So from the equality of functions $\mu_{n-1}(K\cap\xi^\perp)=\mu_{n-1}(L\cap\xi^\perp)$ on S^{n-1} we get the equality of those functions on $\mathbb{R}^n$:

$$\Big[|x|^{-n+1}\int\limits_0^{\frac{|x|}{\|x\|_K}} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right)dt\Big]^\wedge(\xi)=\Big[|x|^{-n+1}\int\limits_0^{\frac{|x|}{\|x\|_L}} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right)dt\Big]^\wedge(\xi).$$

Applying the inverse Fourier transform to both sides of the latter equation we get:

$$\int\limits_0^{|x|/||x||_K} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right)dt=\int\limits_0^{|x|/||x||_L} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right)dt,\quad\forall x\in\mathbb{R}^n,$$

which, together with monotonicity of the function $\int_0^y t^{n-2}f_{n-1}(\frac{tx}{|x|})dt$, for $y\in\mathbb{R}^+$ (because $f_{n-1}(x)>0$ everywhere except for a countable set of points $x\in\mathbb{R}^n$), gives $||x||_K=||x||_L$.

□

3. Proofs of Theorems 1 and 2

We would like to start with the following elementary inequality:

Lemma 1. (Elementary inequality)

$$\int_0^a t^{n-1}\alpha(t)dt - a\frac{\alpha(a)}{\beta(a)}\int_0^a t^{n-2}\beta(t)dt$$
$$\leq \int_0^b t^{n-1}\alpha(t)dt - a\frac{\alpha(a)}{\beta(a)}\int_0^b t^{n-2}\beta(t)dt. \qquad (6)$$

for all $a, b > 0$ *and* $\alpha(t), \beta(t)$ *being positive continuous functions on* $(0, \max\{a, b\}]$, *such* $t\frac{\alpha(t)}{\beta(t)}$ *is nondecreasing on* $(0, \max\{a, b\}]$.

Proof : The inequality (6) is equivalent to

$$a\frac{\alpha(a)}{\beta(a)}\int_a^b t^{n-2}\beta(t)dt \leq \int_a^b t^{n-1}\alpha(t)dt.$$

But

$$a\frac{\alpha(a)}{\beta(a)}\int_a^b t^{n-2}\beta(t)dt = \int_a^b t^{n-1}\alpha(t)\left(a\frac{\alpha(a)}{\beta(a)}\right)\left(t\frac{\alpha(t)}{\beta(t)}\right)^{-1}dt \leq \int_a^b t^{n-1}\alpha(t)dt.$$

Note that the latter inequality does not require $a \leq b$.

□

Before proving Theorems 1 and 2 we need to state a version of Parseval's identity on the sphere and to remind a few facts concerning positive definite homogeneous distributions.

Suppose that $f(x)$ is a continuous on $\mathbb{R}^n \setminus \{0\}$ function, which is a positive definite homogeneous distribution of degree $-k$, where $1 \leq k \leq n-1$, $k \in \mathbb{N}$. Then the Fourier transform of $f(x)$ is a tempered measure γ on $\mathbb{R}^n$ (see Section 2) which is a homogeneous distribution of degree $-n+k$. Writing this measure in the spherical coordinates (see [K6], Lemma 1) we can find a measure γ_0 on S^{n-1} so that for every even test function ϕ

$$\langle \hat{f}, \phi\rangle = \langle \gamma, \phi\rangle = \int_{S^{n-1}} d\gamma_0(\theta)\int_0^\infty \phi(r\theta)dr.$$

Proposition 1. (Koldobsky, [K5]**)** *Fix* $k \in \mathbb{N}$, $1 \leq k \leq n-1$. *Let* f *and* g *be two even functions on* $\mathbb{R}^n$, *homogeneous of degrees* $-k$ *and* $-n+k$ *respectively. Suppose that* f *represents a positive definite distribution on* $\mathbb{R}^n$ *and* γ_0 *is the measure on* S^{n-1} *defined above. Also assume that* $g \in C^{k-1}(\mathbb{R}^n \setminus \{0\})$ *then*

$$\int_{S^{n-1}} \hat{g}(\theta)d\gamma_0(\theta) = (2\pi)^n \int_{S^{n-1}} g(\theta)f(\theta)d\theta.$$

Remark: It is crucial that the sum of degrees of homogeneity of the functions f and g is equal to $-n$. This is one of the reasons for the choice of degrees of homogeneity in the conditions (2) and (3) from Theorems 1 and 2 and in the formula from Theorem 3.

Proof of Theorem 1: Consider symmetric star bodies K and L in $\mathbb{R}^n$, such that

$$\mu_{n-1}(K\cap\xi^\perp)\le\mu_{n-1}(L\cap\xi^\perp),\ \ \forall\xi\in S^{n-1}. \tag{7}$$

We apply Theorem 3 to get an analytic form of (7):

$$\left[|x|^{-n+1}\int_0^{\frac{|x|}{\|x\|_K}} t^{n-2}f_{n-1}\left(\frac{tx}{|x|}\right)dt\right]^\wedge(\xi)\le\left[|x|^{-n+1}\int_0^{\frac{|x|}{\|x\|_L}} t^{n-2}f_{n-1}\left(\frac{tx}{|x|}\right)dt\right]^\wedge(\xi).$$

Next we integrate the latter inequality over S^{n-1} with respect to the measure γ_0 corresponding to a positive definite homogeneous of degree -1 distribution (2):

$$\int_{S^{n-1}}\left[|x|^{-n+1}\int_0^{\frac{|x|}{\|x\|_K}} t^{n-2}f_{n-1}\left(\frac{tx}{|x|}\right)dt\right]^\wedge(\xi)\,d\gamma_0(\xi)$$
$$\le\int_{S^{n-1}}\left[|x|^{-n+1}\int_0^{\frac{|x|}{\|x\|_L}} t^{n-2}f_{n-1}\left(\frac{tx}{|x|}\right)dt\right]^\wedge(\xi)\,d\gamma_0(\xi). \tag{8}$$

Applying the spherical Parseval identity (Proposition 1, with $k=1$) we get:

$$\int_{S^{n-1}}\|x\|_K^{-1}\frac{f_n(\frac{x}{\|x\|_K})}{f_{n-1}(\frac{x}{\|x\|_K})}\int_0^{\|x\|_K^{-1}} t^{n-2}f_{n-1}(tx)dtdx$$
$$\le\int_{S^{n-1}}\|x\|_K^{-1}\frac{f_n(\frac{x}{\|x\|_K})}{f_{n-1}(\frac{x}{\|x\|_K})}\int_0^{\|x\|_L^{-1}} t^{n-2}f_{n-1}(tx)dtdx. \tag{9}$$

Now we apply Lemma 1, with $a=\|x\|_K^{-1}$, $b=\|x\|_L^{-1}$, $\alpha(t)=f_n(tx)$ and $\beta(t)=f_{n-1}(tx)$ (note that from condition (1) it follows that $t\alpha(t)/\beta(t)$ is nondecreasing) to get

$$\int_0^{\|x\|_K^{-1}} t^{n-1}f_n(tx)dt-\|x\|_K^{-1}\frac{f_n(\frac{x}{\|x\|_K})}{f_{n-1}(\frac{x}{\|x\|_K})}\int_0^{\|x\|_K^{-1}} t^{n-2}f_{n-1}(tx)dt$$
$$\le\int_0^{\|x\|_L^{-1}} t^{n-1}f_n(tx)dt-\|x\|_K^{-1}\frac{f_n(\frac{x}{\|x\|_K})}{f_{n-1}(\frac{x}{\|x\|_K})}\int_0^{\|x\|_L^{-1}} t^{n-2}f_{n-1}(tx)dt,\ \forall x\in S^{n-1}.$$

Integrating over S^{n-1} we get

$$\int_{S^{n-1}}\int_0^{\|x\|_K^{-1}} t^{n-1}f_n(tx)dtdx-\int_{S^{n-1}}\|x\|_K^{-1}\frac{f_n(\frac{x}{\|x\|_K})}{f_{n-1}(\frac{x}{\|x\|_K})}\int_0^{\|x\|_K^{-1}} t^{n-2}f_{n-1}(tx)dtdx \tag{10}$$

$$\leq \int_{S^{n-1}}\int_0^{\|x\|_L^{-1}} t^{n-1}f_n(tx)dtdx-\int_{S^{n-1}}\|x\|_K^{-1}\frac{f_n(\frac{x}{\|x\|_K})}{f_{n-1}(\frac{x}{\|x\|_K})}\int_0^{\|x\|_L^{-1}} t^{n-2}f_{n-1}(tx)dtdx.$$

Adding equations (9) and (10) we get

$$\int_{S^{n-1}}\int_0^{\|x\|_K^{-1}} t^{n-1}f_n(tx)dtdx \leq \int_{S^{n-1}}\int_0^{\|x\|_L^{-1}} t^{n-1}f_n(tx)dtdx,$$

which is exactly $\mu_n(K)\leq\mu_n(L)$.

□

The next proposition is to show that convexity is preserved under small perturbations, which are needed in the proof of Theorem 2.

Proposition 2. *Consider an infinitely smooth origin symmetric convex body L with positive curvature and even functions $f_{n-1}, g\in C^2(R^n\setminus\{0\})$, such that f_{n-1} is strictly positive on $\mathbb{R}^n\setminus\{0\}$. For $\varepsilon>0$ define a star body D by an equation for its radial function $\|x\|_D^{-1}$:*

$$\int_0^{\|x\|_D^{-1}} t^{n-2}f_{n-1}(tx)dt=\int_0^{\|x\|_L^{-1}} t^{n-2}f_{n-1}(tx)dt-\varepsilon g(x),\ \ \forall x\in S^{n-1}.$$

Then if ε is small enough the body D is convex.

Proof : For small enough ε, define a function $\alpha_\varepsilon(x)$ on S^{n-1} such that

$$\int_0^{||x||_L^{-1}} t^{n-2}f_{n-1}(tx)dt-\varepsilon g(x)=$$

$$\int_0^{||x||_L^{-1}-\alpha_\varepsilon(x)} t^{n-2}f_{n-1}(tx)dt,\ \ \forall x\in S^{n-1}. \tag{11}$$

Using monotonicity of $\int_0^y t^{n-2}f_{n-1}(tx)dt$, for $y\in\mathbb{R}^+$ ($f_{n-1}(tx)>0$, for $tx\in\mathbb{R}^n\setminus\{0\}$), we get

$$||x||_D^{-1}=||x||_L^{-1}-\alpha_\varepsilon(x),\ \ \forall x\in S^{n-1}. \tag{12}$$

Moreover, $f_{n-1}(x)$, $x\in S^{n-1}$, and its partial derivatives of order two are bounded for finite values of x, so we get, from (11), that $\alpha_\varepsilon(x)$ and its first and second derivatives converge uniformly to 0 (for $x\in S^{n-1}$, as $\varepsilon\to 0$).

Using that L is convex with positive curvature, one can choose a small enough ε so that the body D is convex (with positive curvature).

$\square$

We will also need two following facts, which are corollaries of Theorem 1 in [GKS] (see also [K5]):

Fact 1: If g is positive, symmetric, homogeneous function of degree -1 such that $g(x) \in C^{(n-2)}(\mathbb{R}^n \setminus \{0\})$, then $\hat{g}(x)$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$.

Fact 2: If f is positive, symmetric, continuous, homogeneous function of degree $-n+1$ on $\mathbb{R}^n \setminus \{0\}$, then $\hat{f}(x)$ is a positive continuous function on $\mathbb{R}^n \setminus \{0\}$.

Proof of Theorem 2: First we will use that the function (3) is in $C^{n-2}(\mathbb{R}^n \setminus \{0\})$, and Fact 1 from above, to claim that

$$\left(\|x\|_L^{-1} \frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)} \right)^{\wedge}$$

is a continuous function on S^{n-1}.

This function does not represent a positive definite distribution so it must be negative on some open symmetric subset Ω of S^{n-1}. Consider positive even function supported $h \in C^\infty(S^{n-1})$ in Ω. Extend h to a homogeneous function $h(\theta)r^{-1}$ of degree -1. Then the Fourier transform of h is a homogeneous function of degree $-n+1$: $\widehat{h(\theta)r^{-1}} = g(\theta)r^{-n+1}$.

For $\varepsilon > 0$, we define a body D by

$$|x|^{-n+1} \int_0^{\frac{|x|}{\|x\|_D}} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt$$

$$= |x|^{-n+1} \int_0^{\frac{|x|}{\|x\|_L}} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt - \varepsilon g\left(\frac{x}{|x|}\right) |x|^{-n+1}.$$

By Proposition 2 one can choose a small enough ε so that the body D is convex.

Since $h \geq 0$, we have

$$\mu_{n-1}(D \cap \xi^\perp) = \frac{1}{\pi}\Big[|x|^{-n+1} \int_0^{|x|/||x||_D} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt\Big]^{\wedge}(\xi)$$

$$= \frac{1}{\pi}\Big[|x|^{-n+1} \int_0^{|x|/||x||_L} t^{n-2} f_{n-1}\left(\frac{tx}{|x|}\right) dt\Big]^{\wedge}(\xi) - \frac{(2\pi)^n \varepsilon h(\xi)}{\pi}$$

$$\leq \mu_{n-1}(L \cap \xi^\perp).$$

On the other hand, the function h is positive only where

$$\left(\|x\|_L^{-1}\frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)}\right)^{\wedge}(\xi)$$

is negative so

$$\left(\|x\|_L^{-1}\frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)}\right)^{\wedge}(\xi)\left(|x|^{-n+1}\int_0^{|x|/\|x\|_D} t^{n-2}f_{n-1}\left(\frac{tx}{|x|}\right)dt\right)^{\wedge}(\xi)$$

$$= \left(\|x\|_L^{-1}\frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)}\right)^{\wedge}(\xi)\left(|x|^{-n+1}\int_0^{|x|/\|x\|_L} t^{n-2}f_{n-1}\left(\frac{tx}{|x|}\right)dt\right)^{\wedge}(\xi)$$

$$-(2\pi)^n\left(\|x\|_L^{-1}\frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)}\right)^{\wedge}(\xi)\varepsilon h(\xi)$$

$$\geq \left(\|x\|_L^{-1}\frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)}\right)^{\wedge}(\xi)\left(|x|^{-n+1}\int_0^{|x|/\|x\|_L} t^{n-2}f_{n-1}\left(\frac{tx}{|x|}\right)dt\right)^{\wedge}(\xi).$$

Note that the latter inequality is strict on Ω. Integrate the latter inequality over S^{n-1} and apply the spherical Parseval identity (Proposition 1, with $k = n-1$ and Fact 2 from above). Finally, the same computations (based on Lemma 1) as in the proof of Theorem 1 give

$$\mu_n(D) > \mu_n(L).$$

□

4. Applications

Corollary 2. *Let $f_n(x) = f_{n-1}(x)$ be equal even non-negative continuous functions:*

- *the answer to the BPGM problem is affirmative for $n \leq 4$,*
- *if f_{n-1} is positive twice differentiable function on $\mathbb{R}^n \setminus \{0\}$, then the answer to the BPGM problem is negative for $n \geq 5$.*

Proof : In this case $tf_n(tx)/f_{n-1}(tx) = t$ is an nondecreasing function, so we may apply Theorems 1 and 2. First note that

$$\|x\|_K^{-1}\frac{f_n\left(\frac{x}{\|x\|_K}\right)}{f_{n-1}\left(\frac{x}{\|x\|_K}\right)} = ||x||_K^{-1}.$$

Thus we may use the fact that for any origin symmetric convex body K in $\mathbb{R}^n$, $n \leq 4$, $||\cdot||_K^{-1}$ represents a positive definite distribution (see [GKS], [K9]) to give the affirmative answer to BPGM in this case.

For $n \geq 5$, we note that there is an infinitely smooth, symmetric, convex body $L \subset \mathbb{R}^n$ with positive curvature and such that $|| \cdot ||_L^{-1}$ is not positive definite (see [GKS], [K9]) and thus

$$\|x\|_L^{-1} \frac{f_n\left(\frac{x}{\|x\|_L}\right)}{f_{n-1}\left(\frac{x}{\|x\|_L}\right)} = ||x||_L^{-1}$$

is in $C^\infty(\mathbb{R}^n \setminus \{0\})$ and does not represent a positive definite distribution. Also f_{n-1} is positive twice differentiable function on $\mathbb{R}^n \setminus \{0\}$, and thus we apply Theorem 2 to finish the proof.

□

Remark 1: The answers for the original Busemann-Petty problem and the Busemann-Petty problem for Gaussian Measures are particular cases of Corollary 2, with $f_n(x) = 1$ and $f_n(x) = e^{-|x|^2/2}$ respectively.
Remark 2: Note that for any $r > 0$ and $n \geq 5$ there is an infinitely smooth, symmetric, convex body $L \subset \mathbb{R}^n$ with positive curvature and such that $||\cdot||_L^{-1}$ is not positive definite and $L \subset rB_2^n$, where B_2^n is the standard Euclidean ball. Thus in the second part of Corollary 2 it is enough to assume that f_{n-1} is positive and twice differentiable on $rB_2^n \setminus \{0\}$ for some $r > 0$. Also note, it is necessary to have an assumption on positivity of functions f_{n-1}, otherwise one may consider $f_{n-1} = f_n = 0$ for which the answer to BPGM is trivially true in all dimensions.

Lemma 2. *Consider a symmetric star body $M \subset \mathbb{R}^n$ such that $\|x\|_M^{-1}$ is positive definite, then for any symmetric star bodies $K, L \subset \mathbb{R}^n$ such that, for every $\xi \in S^{n-1}$,*

$$\int_{K\cap\xi^\perp} \|x\|_M \, dx \leq \int_{L\cap\xi^\perp} \|x\|_M \, dx \tag{13}$$

we have

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L).$$

Proof : This follows from Theorem 1 with $f_n = 1$, $f_{n-1} = \|x\|_M$. In this case,

$$\|x\|_K^{-1} \frac{f_n\left(\frac{x}{\|x\|_K}\right)}{f_{n-1}\left(\frac{x}{\|x\|_K}\right)} = \|x\|_K^{-1} \frac{1}{\|\frac{x}{\|x\|_K}\|_M} = ||x||_M^{-1}.$$

□

Lemma 3. *Consider a symmetric star body $M \subset \mathbb{R}^n$ such that $\|x\|_M^{-1}$ is positive definite. Then for any symmetric star bodies $K, L \subset \mathbb{R}^n$ such that, for every $\xi \in S^{n-1}$,*

$$\mathrm{Vol}_{n-1}(K \cap \xi^\perp) \leq \mathrm{Vol}_{n-1}(L \cap \xi^\perp), \tag{14}$$

we have

$$\int_K \|x\|_M^{-1} \, dx \leq \int_L \|x\|_M^{-1} \, dx$$

Proof : This theorem follows by the same argument as in Lemma 2, but with $f_n = \|x\|_M^{-1}$, $f_{n-1} = 1$.

$\square$

Remark: It follows from Theorem 2 that Lemmas 2 and 3 are not true (even, with additional convexity assumption on K and L) if M is smooth and $\|x\|_M^{-1}$ is not positive definite.

Before presenting another application of Theorem 1, we would like to remind the statement of the lower dimensional Busemann-Petty problem. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, fix some $k \in \mathbb{N}$ such that $1 \le k \le n-1$. Suppose that for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$,

$$\mathrm{Vol}_{n-k}(K \cap H) \le \mathrm{Vol}_{n-k}(L \cap H).$$

Does it follow that

$$\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L)?$$

It was proved by Bourgain and Zhang [BZ] that the answer to the lower dimensional Busemann-Petty problem is negative if the dimension of sections $n-k > 3$. Another proof of the same fact was given by Koldobsky [K7]. The problem is still open for $n-k = 2, 3$ (see [BZ], [K7] and [RZ] for more details)

We will use Theorem 1 and ideas from Lemmas 2 and 3 to show some results on a lower dimensional version of the BPGM.

Theorem 4. *Consider a symmetric star body $M \subset \mathbb{R}^n$ such that $\|x\|_M^{-1}$ is positive definite, then for any symmetric star bodies $K, L \subset \mathbb{R}^n$, and $1 \le k < n$ such that, for every $H \in G(n, n-k)$*

$$\int_{K\cap H} ||x||_M^k \, dx \le \int_{L\cap H} ||x||_M^k \, dx,$$

we have

$$\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L).$$

Proof : It was proved in [GW] that every hyperplane section of an intersection body is also an intersection body. Using the relation between positive definite distributions and intersection bodies we get that if $\|x\|_M^{-1}$ is a positive definite distribution then the restriction of $\|x\|_M^{-1}$ to a subspace $F \in G(n, n-i+1)$, $i \in N$, $1 \le i \le n$ is also a positive definite distribution. Thus we may apply Theorem 1 with functions $f_{i-1}(x) = ||x||_M^{n-i+1}$ and $f_i(x) = ||x||_M^{n-i}$. Indeed, in this case

$$\|x\|_K^{-1} \frac{f_i\left(\frac{x}{\|x\|_K}\right)}{f_{i-1}\left(\frac{x}{\|x\|_K}\right)} = \|x\|_K^{-1} \frac{\|\frac{x}{\|x\|_K}\|_M^{n-i}}{\|\frac{x}{\|x\|_K}\|_M^{n-i+1}} = ||x||_M^{-1}$$

is a positive definite distribution. So, from Theorem 1, we get that if for every $H \in G(n, n-i)$

$$\int_{K\cap H} ||x||_M^i \, dx \leq \int_{L\cap H} ||x||_M^i \, dx,$$

then for every $F \in G(n, n-i+1)$

$$\int_{K\cap F} ||x||_M^{i-1} \, dx \leq \int_{L\cap F} ||x||_M^{i-1} \, dx.$$

We iterate this procedure for $i = k, k-1, \dots, 1$ to finish the proof.

□

We can present a different version of Theorem 4, in a special cases of $n-k = \{2,3\}$ and K is a convex symmetric body:

Corollary 3. *Consider a symmetric star body $M \subset \mathbb{R}^n$, $n \geq 4$, such that $\|x\|_M^{-1}$ is positive definite. Fix k such that $n-k \in \{2,3\}$, then for any convex symmetric bodies $K, L \subset \mathbb{R}^n$, such that, for every $H \in G(n, n-k)$*

$$\int_{K\cap H} ||x||_M^{n-4} \, dx \leq \int_{L\cap H} ||x||_M^{n-4} \, dx, \tag{15}$$

we have

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L).$$

Proof : We use the same iteration procedure as in Theorem 4. The difference is only in the first steps of iteration procedure. For example, from subspaces of dimension 2 to subspaces of dimension 3 we may use Corollary 2 with $f_2(x) = f_3(x) = ||x||_M^{n-4}$, thus this iteration does not change the power of $||x||_M$. We use the same idea to iterate from dimension 3 to 4.

□

Remark: Note that if $n-k = 2$, then Corollary 3 is still true with the power $n-3$, instead of $n-4$ in (15). Corollary 3 is a generalization of a result of Koldobsky ([K7], Theorem 8; see also [RZ]), where the case of $n-k = 3$ and $M = B_2^n$ was considered.

Let $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ and $B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}$.

Corollary 4. *Consider $p \in (0,2]$ and $n \geq 2$. Then for any symmetric star bodies $K, L \subset \mathbb{R}^n$ such that*

$$\int_{K\cap\xi^\perp} \|x\|_p \, dx \leq \int_{L\cap\xi^\perp} \|x\|_p \, dx \tag{16}$$

for every $\xi \in S^{n-1}$, we have

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L).$$

Proof : This follows from Lemma 2 and the fact that $\|x\|_p^{-1}$ is positive definite for $p \in (0,2]$ (see [K4], [K9]).

□

Corollary 5. *Consider $p > 2$ and $n \geq 5$. Then there are convex symmetric bodies $K, L \subset \mathbb{R}^n$ such that*

$$\int_{K\cap\xi^\perp} \|x\|_p \, dx \leq \int_{L\cap\xi^\perp} \|x\|_p \, dx, \tag{17}$$

but

$$\mathrm{Vol}_n(K) > \mathrm{Vol}_n(L).$$

Proof : Note that $\|x\|_p^{-1}$ does not represent a positive definite distribution when $p > 2$ and $n \geq 5$ (see [K4], [K9]). To prove the above corollary we will use the proof of Theorem 2. Indeed, it was proved in [K3] that $\left(\|x\|_p^{-1}\right)^\wedge$, $p > 2$ is a locally integrable sign changing function on $\mathbb{R}^n$, thus we may find a set $\Omega \in S^{n-1}$ and function h as in the proof of Theorem 2. Next we consider an infinitely smooth origin symmetric convex body L in $\mathbb{R}^n$ with positive curvature (for example $L = B_2^n$). Note that $\|x\|_p^{-1}$, $p > 2$ is twice differentiable positive on $\mathbb{R}^n \setminus \{0\}$ and thus Proposition 2 can be applied and convex body D can be constructed. To finish the proof we apply a special version of the spherical Parseval identity for $\|x\|_p^{-1}$, which follows from Proposition 1 with an additional approximation argument (see Appendix, Lemma 5).

□

Next we would like to use Lemma 2 to give a lower bound on the integral over a hyperplane section of the convex body. This result is motivated by the slicing problem (see [MP] for more details) however instead of lower estimates for volumes of sections we get estimates for integrals of certain norms over sections.

Lemma 4. *Consider a symmetric star body $M \subset \mathbb{R}^n$ such that $\|x\|_M^{-1}$ is positive definite. Then for any star body $K \subset \mathbb{R}^n$ there exists a direction $\xi \in S^{n-1}$ such that*

$$\int_{K\cap\xi^\perp} \|x\|_M dx \geq \frac{n-1}{n}\frac{\mathrm{Vol}_{n-1}(M\cap\xi^\perp)}{\mathrm{Vol}_n(M)}\mathrm{Vol}_n(K).$$

Proof : Assume it is not true, then

$$\int_{K\cap\xi^\perp} \|x\|_M dx < \frac{n-1}{n}\frac{\mathrm{Vol}_{n-1}(M\cap\xi^\perp)}{\mathrm{Vol}_n(M)}\mathrm{Vol}_n(K), \ \ \forall\xi \in S^{n-1}.$$

Also note that for $L \subset \mathbb{R}^{n-1}$

$$\int_L \|x\|_L dx = \frac{1}{n}\int_{S^{n-1}} \|\theta\|_L^{-n} d\theta = \frac{n-1}{n}\mathrm{Vol}_{n-1}(L).$$

Applying the latter equality with $L = M \cap \xi^\perp$ we get

$$\int_{K\cap\xi^\perp} \|x\|_M dx < \frac{\mathrm{Vol}_n(K)}{\mathrm{Vol}_n(M)}\int_{M\cap\xi^\perp} \|x\|_M dx, \ \ \forall\xi \in S^{n-1}.$$

Let $r^n = \frac{\mathrm{Vol}_n(K)}{\mathrm{Vol}_n(M)}$, then

$$\int_{K\cap\xi^\perp} \|x\|_{rM} dx < \int_{rM\cap\xi^\perp} \|x\|_{rM} dx, \ \ \forall \xi \in S^{n-1}.$$

Thus

$$\mathrm{Vol}_n(K) < \mathrm{Vol}(rM),$$

or

$$\mathrm{Vol}_n(K) < \frac{\mathrm{Vol}_n(K)}{\mathrm{Vol}_n(M)} \mathrm{Vol}_n(M)$$

which gives a contradiction.

□

Corollary 6. *For any $p \in [1,2]$ and any symmetric star body K in $\mathbb{R}^n$ there exists a direction $\xi \in S^{n-1}$ such that*

$$\int_{K\cap\xi^\perp} \|x\|_p dx \geq c_p n^{1/p} \mathrm{Vol}_n(K),$$

where c_p is a constant depending on p only.

Proof : Again $\|x\|_p^{-1}$, $p \in [1,2]$ is a positive definite distribution (see [K4], [K9]) and thus we may apply Lemma 4 (or Corollary 4) to get that there exists $\xi \in S^{n-1}$ such that

$$\int_{K\cap\xi^\perp} \|x\|_p dx \geq \frac{n-1}{n} \frac{\mathrm{Vol}_{n-1}(B_p^n \cap \xi^\perp)}{\mathrm{Vol}_n(B_p^n)} \mathrm{Vol}_n(K).$$

Next we use that B_p^n, $p \in [1,2]$ is in the isotropic position (see [MP] for definition), and the isotropic constant $L_{B_p^n} \leq c$ (see [Sc]), thus the ratio of volume of different hyperplane sections is bounded by two universal constants (see [H], [Bo1] or [MP], Corollary 3.2):

$$c \leq \frac{\mathrm{Vol}_{n-1}(B_p^n \cap \xi^\perp)}{\mathrm{Vol}_{n-1}(B_p^n \cap \nu^\perp)} \leq C, \ \ \forall \xi, \nu \in S^{n-1},$$

and

$$c\mathrm{Vol}_{n-1}(B_p^{n-1}) \leq \mathrm{Vol}_{n-1}(B_p^n \cap \xi^\perp) \leq C\mathrm{Vol}_{n-1}(B_p^{n-1}), \ \ \forall \xi \in S^{n-1}.$$

Applying

$$\mathrm{Vol}_n(B_p^n) = \frac{[2\Gamma(1+\frac{1}{p})]^n}{\Gamma(1+\frac{n}{p})},$$

we get

$$\frac{\mathrm{Vol}_{n-1}(B_p^{n-1})}{\mathrm{Vol}_n(B_p^n)} = \frac{1}{2\Gamma(1+\frac{1}{p})} \frac{\Gamma(1+\frac{n}{p})}{\Gamma(1+\frac{n-1}{p})} \geq c_p n^{1/p}.$$

□

Remark: The above corollary can be generalized to the case of any body M, with positive definite $\|x\|_M^{-1}$ such that M is in the isotropic position.

Corollary 7. *For any convex symmetric body $K \in \mathbb{R}^n$, there are vectors $\xi, \nu \in S^{n-1}$ such that $\xi \neq \pm\nu$ and*

$$\mathrm{Vol}_{n-1}(K \cap \xi^\perp) \geq c\sqrt{\mathrm{Vol}_{n-2}(K \cap \{\xi,\nu\}^\perp)\mathrm{Vol}_n(K)},$$

where $\{\xi,\nu\}^\perp$ is the subspace of codimension 2 orthogonal to ξ and ν.

Proof : From Corollary 6 (with $p = 1$) we get that there exists a direction $\xi \in S^{n-1}$ such that

$$\int_{K\cap\xi^\perp} \sum_{i=1}^{n} |x_i|dx \geq cn\mathrm{Vol}_n(K). \tag{18}$$

By continuity of the volume measure, we may assume that $\xi \notin \{\pm e_i\}_{i=1}^n$, where $\{e_i\}_{i=1}^n$ is the standard basis in $\mathbb{R}^n$. From the inequality (18) we get that there exists $j \in \{1, \ldots, n\}$ such that

$$\int_{K\cap\xi^\perp} |x_j|dx \geq c\mathrm{Vol}_n(K). \tag{19}$$

Next we apply the "inverse Holder" inequality ([MP], Corollary 2.7; see also [GrM]): for any symmetric convex body L in $\mathbb{R}^{n-1}$ and unit vector $\theta \in S^{n-2}$

$$\int_L |x\cdot\theta|dx \leq c\frac{\mathrm{Vol}^2_{n-1}(L)}{\mathrm{Vol}_{n-2}(L\cap\theta^\perp)},$$

Applying the latter inequality to $L = K \cap \xi^\perp$ and $\theta = e_j$ we get

$$\int_{K\cap\xi^\perp} |x_j|dx \leq c\frac{\mathrm{Vol}^2_{n-1}(K\cap\xi^\perp)}{\mathrm{Vol}_{n-1}(K\cap\{\xi,e_j\}^\perp)}. \tag{20}$$

Combining inequalities (19) and (20) we finish the proof.

□

Appendix: A version of spherical Parseval's formula for $\|x\|_p$.

Lemma 5. *Let f be a continuous even function on $\mathbb{R}^n \setminus \{0\}$, which is homogenous of degree $-n+1$,*

$$(2\pi)^n \int\limits_{S^{n-1}} \|\theta\|_p^{-1} f(\theta)d\theta = \int\limits_{S^{n-1}} \widehat{\|x\|_p^{-1}}(\theta)\hat{f}(\theta)d\theta.$$

Proof : It was proved in [K3] that $\left(\|\cdot\|_p^{-1}\right)^\wedge$, is a locally integrable function, so the left hand side of the above inequity is well-defined and our goal is to show the equality.

We follow [KZ]. Consider the following approximation of $\|x\|_p$

$$\|x\|_{p,\varepsilon} = \Big[\sum_{i=1}^{n}\big(x_i^{2k} + \varepsilon\sum_{j\neq i} x_j^{2k}\big)^{\frac{p}{2k}}\Big]^{\frac{1}{p}},$$

where $\varepsilon > 0$ and k is a fixed positive integer such that $2k > p$.

Clearly, $\|x\|_{p,\varepsilon}$ is a continuous function of ε and $\|x\|_{p,\varepsilon} \in C^\infty(S^n)$. Moreover, $\|x\|_{p,\varepsilon} \to \|x\|_p$, as $\varepsilon \to 0^+$, uniformly with respect to $x \in S^n$.

Thus we may apply spherical Parseval identity (Proposition 1) to get that

$$(2\pi)^n \int\limits_{S^{n-1}} \|\theta\|_{p,\varepsilon}^{-1} f(\theta) d\theta = \int\limits_{S^{n-1}} \widehat{\|x\|_{p,\varepsilon}^{-1}}(\theta) \hat{f}(\theta) d\theta.$$

Obviously, the left-hand side of the latter equality converges to the same integral with $\|x\|_p$ in place of $\|x\|_{p,\varepsilon}$, as $\varepsilon \to 0$. Therefore, our goal is to prove that the same happens in the right-hand side.

Note that $p/k \in (0,2]$, so by Bernstein's Theorem (see [W]) there exists a measure $\mu_{(p/k)}$ on $[0,\infty)$ so that, for every $t \in \mathbb{R}$,

$$e^{-|t|^{\frac{p}{2k}}} = \int_0^\infty e^{-ut} d\mu_{(p/k)}(u).$$

thus for all $x_1, \cdots, x_n$ and ε:

$$e^{-|x_i^{2k} + \varepsilon \sum\limits_{j\neq i} x_j^{2k}|^{\frac{p}{2k}}} = \int_0^\infty e^{-u\left(x_i^{2k} + \varepsilon \sum\limits_{j\neq i} x_j^{2k}\right)} d\mu_{(p/k)}.$$

¿From the definition of the Gamma function, we have that

$$\|x\|^{-1} = \frac{p}{\Gamma(1/p)} \int_0^\infty e^{-t^p \|x\|^p} dt.$$

Thus

$$\|x\|_{p,\varepsilon}^{-1} = \frac{p}{\Gamma(1/p)} \int_0^\infty e^{-\sum\limits_{i=1}^n \left(t^{2k}\left(x_i^{2k} + \varepsilon \sum\limits_{j\neq i} x_j^{2k}\right)\right)^{\frac{p}{2k}}} dt$$

$$= \frac{p}{\Gamma(\frac{1}{p})} \int_0^\infty \int_{\mathbb{R}^n} e^{-\sum\limits_{i=1}^n u_i \left(t^{2k}\left(x_i^{2k} + \varepsilon \sum\limits_{j\neq i} x_j^{2k}\right)\right)} d\mu_{(p/k)}(u) dt$$

Then

$$\langle \widehat{\|x\|_{p,\varepsilon}^{-1}}, \ \phi \rangle = \langle \|x\|_{p,\varepsilon}^{-1}, \ \widehat{\phi} \rangle$$

$$= \frac{p}{\Gamma(1/p)} \int\limits_{\mathbb{R}^n} \Big[\int\limits_0^\infty \int\limits_{\mathbb{R}^n} e^{-\sum\limits_{i=1}^n u_i \left(t^{2k}\left(x_i^{2k} + \varepsilon \sum\limits_{j\neq i} x_j^{2k}\right)\right)} d\mu_{(p/k)}(u) dt \Big] \widehat{\phi(x)} dx$$

$$= \frac{p}{\Gamma(1/p)} \int\limits_0^\infty \int\limits_{\mathbb{R}^n} \Big[\int\limits_{\mathbb{R}^n} e^{-\sum\limits_{i=1}^n u_i \left(t^{2k}\left(x_i^{2k} + \varepsilon \sum\limits_{j\neq i} x_j^{2k}\right)\right)} \widehat{\phi(x)} dx \Big] d\mu_{(p/k)}(u) dt$$

$$= \frac{p}{\Gamma(1/p)} \int\limits_0^\infty \int\limits_{\mathbb{R}^n} \Big[\int\limits_{\mathbb{R}^n} \Big(e^{-\sum\limits_{i=1}^n u_i \left(t^{2k}\left(x_i^{2k} + \varepsilon \sum\limits_{j\neq i} x_j^{2k}\right)\right)} \Big)^\wedge \phi(x) dx \Big] d\mu_{(p/k)}(u) dt.$$

Note that $\sum\limits_{i=1}^n u_i\big(t^{2k}\big(x_i^{2k}+\varepsilon\sum\limits_{j\neq i}x_j^{2k}\big)\big)=\sum\limits_{i=1}^n \big(t^{2k}\big(u_i+\varepsilon\sum\limits_{j\neq i}u_j\big)\big)x_i^{2k}$.

Denote $U_i=\big(u_i+\varepsilon\sum\limits_{j\neq i}u_j\big)^{\frac{1}{2k}}$, then

$$= \frac{p}{\Gamma(1/p)}\int_0^\infty\int_{\mathbb{R}^n}\Big[\int\limits_{\mathbb{R}^n}\Big(\prod_{i=1}^n\frac{\gamma_{2k}\big(\frac{x_i}{tU_i}\big)}{tU_i}\Big)\phi(x)dx\Big]d\mu_{(p/k)}(u)dt,$$

where γ_{2k} is the Fourier transform of function $e^{-|t|^{2k}}$. Thus

$$\widehat{\|x\|_{p,\varepsilon}^{-1}}=\frac{p}{\Gamma(\frac{1}{p})}\int_0^\infty\int_{\mathbb{R}^n}\left(\prod_{i=1}^n\frac{\gamma_{2k}\big(\frac{x}{tU_i}\big)}{tU_i}\right)d\mu_{(p/k)}(u)dt.$$

Using the same method we get

$$\widehat{\|x\|_p^{-1}}=\frac{p}{\Gamma(\frac{1}{p})}\int_0^\infty\int_{\mathbb{R}^n}\left(\prod_{i=1}^n\frac{\gamma_{2k}\left(\frac{x}{tu_i^{1/2k}}\right)}{tu_i^{1/2k}}\right)d\mu_{(p/k)}(u)dt.$$

Note that the function $\gamma_{2k}(x)$, $k\in\mathbb{N}$, decreases exponentially at infinity (see for example [K3], [K11]) and the measure $\mu_q(x)$ has density that decreases at infinity like $|x|^{-1-q/2}$, which makes the function under the integral bounded and continuous. Finally, we use the fact that $\hat{f}$ is a continuous function together with dominated convergence theorem to finish the proof (see [KZ] for more details).

□

References

[Ba] K. Ball, *Cube slicing in* $\mathbb{R}^n$, Proc. Amer. Math. Soc. **97** (1986), 465-473.

[Bo1] J. Bourgain, *On high dimensional maximal functions associated to convex bodies*, Amer. J. Math. **108** (1986), 1467-1476.

[Bo2] J. Bourgain, *On the Busemann-Petty problem for perturbations of the ball*, Geom. Funct. Anal. **1** (1991), 1-13.

[BZ] J. Bourgain, Gaoyong Zhang, *On a generalization of the Busemann-Petty problem,* Convex geometric analysis (Berkeley, CA, 1996), 65-76, Math. Sci. Res. Inst. Publ., 34, Cambridge Univ. Press, Cambridge, (1999).

[BP] H. Busemann, C.M. Petty, *Problems on convex bodies*, Math. Scand. **4** (1956), 88-94.

[Ga1] R.J. Gardner, *Intersection bodies and the Busemann-Petty problem*, Trans. Amer. Math. Soc. **342** (1994), 435-445.

[Ga2] R.J. Gardner, *A positive answer to the Busemann-Petty problem in three dimensions*, Ann. of Math. (2) **140** (1994), 435-447.

[Ga3] R.J. Gardner, *Geometric tomography*, Cambridge Univ. Press, New York, 1995.

[GKS] R.J. Gardner, A. Koldobsky, and Th. Schlumprecht, *An analytic solution of the Busemann-Petty problem on sections of convex bodies*, Annals of Math. **149** (1999), 691-703.

[GS] I.M. Gelfand and G.E. Shilov, *Generalized functions, vol. 1. Properties and operations,* Academic Press, New York, (1964).

[GV] I.M. Gelfand and N. Ya. Vilenkin, *Generalized functions, vol. 4. Applications of harmonic analysis*, Academic Press, New York, 1964.

[Gi] A. Giannopoulos, *A note on a problem of H. Busemann and C. M. Petty concerning sections of symmetric convex bodies*, Mathematika **37** (1990), 239-244.

[GW] P. Goodey, W. Weil, *Intersection bodies and ellipsoids*, Mathematika **42** (1995), no. 2, 295-304.

[GrM] M. Gromov, V.D. Milman *Brunn theorem and a concentration of volume of convex bodies*, GAFA Seminar Notes, Tel Aviv University, Israel 1983-1984, Exp. V., 12pp..

[H] D. Hensley, *Slicing convex bodies-bounds for slice area in terms of the body's covariance,* Proc. Amer. Math. Soc. **79** (1980), 619-625.

[K1] A. Koldobsky, *Inverse formula for the Blaschke-Levy representation*, Houston J. Math. **23** (1997), 95-108.

[K2] A. Koldobsky, *An application of the Fourier transform to sections of star bodies*, Israel J. Math **106** (1998), 157-164.

[K3] A. Koldobsky, *Intersection Bodies in* $\mathbb{R}^4$, Adv. Math. **136** (1998), 1-14.

[K4] A. Koldobsky, *Intersection bodies, positive definite distributions, and the Busemann-Petty problem*, Amer. J. Math. **120** (1998), no. 4, 827-840.

[K5] A. Koldobsky, *A generalization of the Busemann-Petty problem on sections of convex bodies*, Israel J. Math. **110** (1999), 75-91.

[K6] A. Koldobsky, *Positive definite distributions and subspaces of* L_{-p} *with applications to stable processes*, Canad. Math. Bull. 42 (1999), **no. 3**, 344-353.

[K7] A. Koldobsky, *A functional analytic approach to intersection bodies*, Geom. Funct. Anal. **10** (2000), no. 6, 1507-1526.

[K8] A. Koldobsky, *On the derivatives of X-ray functions*, Arch. Math. **79**, (2002), 216–222.

[K9] A. Koldobsky, *Sections of star bodies and the Fourier transform*, Proceedings of the AMS-IMS-SIAM Summer Research Conference in Harmonic Analysis, Mt Holyoke, 2001, Contemp. Math. 320 (2003), 225-248.

[K10] A. Koldobsky, *Comparison of volumes by means of the areas of central sections*, preprint.

[K11] A. Koldobsky, *Fourier analysis and convex geometry*, book preprint.

[KYY] A. Koldobsky, M. Yaskina, V. Yaskin, *Modified Busemann-Petty problem on Sections of convex bodies,* preprint.

[KZ] A. Koldobsky, M. Zymonopoulou, *Extremal sections of complex* l_p*-balls,* $0 < p < 2$*,* Studia Mathematica **159** (2003), 185-194.

[LR] D.G. Larman and C.A. Rogers, *The existence of a centrally symmetric convex body with central sections that are unexpectedly small*, Mathematika **22** (1975), 164-175.

[Lu] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232-261.

[MP] V.D. Milman, A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed* n*-dimensional space*, Geometric aspects of functional analysis (1987–88), Lecture Notes in Math., **1376**, Springer, Berlin, (1989), 64-104.

[Pa] M. Papadimitrakis, *On the Busemann-Petty problem about convex, centrally symmetric bodies in* $\mathbb{R}^n$, Mathematika **39** (1992), 258-266.

[RZ] B. Rubin, Gaoyong Zhang, *Generalizations of the BusemannPetty problem for sections of convex bodies*, J. Funct. Anal., to appear.

[Sc] M. Schmuckenschläger, *Volume of intersections and sections of the unit ball of* l_p^n, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1527-1530.

[W] D.V. Widder, *The Laplace transform,* Princeton University Press, Princeton, NJ, 1941.

[Zh1] Gaoyong Zhang, *Centered bodies and dual mixed volumes,* Trans. A.M.S., **345**, (1994), 777-801.

[Zh2] Gaoyong Zhang, *A positive answer to the Busemann-Petty problem in four dimensions*, Annals of Math. **149** (1999), 535-543.

[Z] A. Zvavitch, *Gaussian Measure of Sections of convex bodies*, Adv. Math., **188/1** (2004), 124-136.

[Zo] V.M. Zolotarev, *One-dimensional stable distributions,* Amer. Math. Soc. Providence, 1986.

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