A metric interpretation of reflexivity for Banach spaces

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Infinite Dimensional Analysis: Celebrating Richard Aron's Work and Impact

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- Two Banach spaces are *isomorphic* if there is a map $T : X \to Y$ that is simultaneously
  - a *linear isomorphism* and
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- Typically, important properties of Banach spaces involve both linear and metric structure.

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Example: X is uniformly convex if for every $\varepsilon > 0$ there exists $\delta > 0$ so that whenever x, y are norm-one vectors with

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- A loosening of uniform convexity: X is super-reflexive if it is isomorphic to a uniformly convex Banach space Y.

(this is not the original definition of super-reflexivity)

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Question 1: Is (P) a *metric property*?

That is, if a Banach space X has (P) and a Banach space Y is *Lipschitz isomorphic* to X, does Y have (P)?

Question 2: Is there a nice or natural characterization of (P) in purely metric terms?

History

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Bourgain's "Ribe Program" (1986): find metric descriptions of *local properties* of Banach spaces.

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For a Banach space X the following are equivalent.

- *X is not super-reflexive.*
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Following Ostrovskii (2003): the binary trees B_n, $n \in \mathbb{N}$, form *a sequence of test spaces* for super-reflexivity.

Metric characterizations of local properties of Banach spaces

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- **Mendel-Naor (2008)**: co-type q.

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- Some *"asymptotic properties"* can be metrically characterized.

Asymptotic super-reflexivity

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Theorem (Szlenk (1968))

Let X be a separable and infinite dimensional Banach space. Then

(i) $\omega \leqslant \mathrm{Sz}(X) \leqslant \omega_1$ *and*

(ii) $\mathrm{Sz}(X) < \omega_1$ *if and only if X^* is separable.*

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Definition

A Banach space X is called *asymptotically super-reflexive* if it is reflexive and $\max\{\mathrm{Sz}(X), \mathrm{Sz}(X^*)\} \leqslant \omega$.

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Theorem (Baudier, Kalton, Lancien (2010))

Let X be a reflexive Banach space. The following are equivalent.

- *X is not asymptotically super-reflexive (i.e. $\max\{\mathrm{Sz}(X), \mathrm{Sz}(X^*)\} > \omega$).*
- *The infinite branching trees T_n uniformly bi-Lipschitzly embed into X.*

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Question: Is reflexivity a metric property?

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Let X be a separable Banach space and Y be a Banach space with the Radon-Nikodým property. If there exists a Lipschitz embedding $\Phi : X \to Y$, then X is linearly isomorphic to a subspace of Y.

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Hence, **yes!** It is preserved via Lipschitz isomorphisms.

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- Find a “nice” metrical characterization of reflexivity.
- Within the class of reflexive Banach spaces, metrically characterize the property

$$\max\{\mathrm{Sz}(X), \mathrm{Sz}(X^*)\} > \alpha$$

for $\omega < \alpha < \omega_1$.

The Schreier families of finite subsets of $\mathbb{N}$

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Definition: They are defined recursively on $\alpha < \omega_1$ as follows:

- $\mathcal{S}_1 = \{F : \#F \leqslant \min(F)\} \cup \{\varnothing\}$.
- $\mathcal{S}_{\alpha+1} = \left\{ \bigcup_{j=1}^{n} A_j : A_j \in \mathcal{S}_\alpha, n \leq A_1 < A_2 < \cdots < A_n \right\}$
- If $\alpha < \omega$ is limit ordinal choose a sequence $\lambda(\alpha, n) \nearrow \alpha$, and set

  $\mathcal{S}_\alpha = \bigcup_{n\in\mathbb{N}} \{A \in \mathcal{S}_{\lambda(\alpha,n)} : \min(A) \geqslant n\}$

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- *Each $\mathcal{S}_\alpha$ is a tree* when endowed with the extension relation.

A metrical characterization of reflexivity

- On each set $\mathcal{S}_\alpha$ we shall define two metrics $d_{\infty,\alpha}$ and $d_{1,\alpha}$.

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Theorem (M. - Schlumprecht)

Let X be a separable Banach space. The following are equivalent.

- *X is non-reflexive.*
- *for all $\alpha < \omega_1$ there exist $0 < c \leqslant C$ and $\Phi : \mathcal{S}_\alpha \to X$ so that*

$$c d_{\infty,\alpha}(A, B) \leqslant \big\|\Phi(A) - \Phi(B)\big\| \leqslant C d_{1,\alpha}(A, B) \text{ for all } A, B \text{ in } \mathcal{S}_\alpha.$$

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Such a map Φ is called a *"lower-$d_{\infty,\alpha}$ - upper-$d_{1,\alpha}$ embedding of $\mathcal{S}_\alpha$"*.

A metrical characterization of the Szlenk index for reflexive spaces

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Theorem (M. - Schlumprecht)

Let X be a separable and reflexive Banach space and $\alpha < \omega_1$ with $\alpha = \omega^\alpha$. The following are equivalent.

- $\max\{\mathrm{Sz}(X), \mathrm{Sz}(X^*)\} > \alpha$.
- $(\mathcal{S}_\alpha, d_{1,\alpha})$ *embeds bi-Lipschitzly into* X.

Questions

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Question: Does the family $(\mathcal{S}_\alpha, d_{1,\alpha})_{\alpha<\omega}$ form a family of test spaces for spaces having non-separable second dual in the class of separable Banach spaces?

That is, does a separable Banach space X have non-separable second dual if and only if $(\mathcal{S}_\alpha, d_{1,\alpha})$ embeds into X for all $\alpha < \omega_1$.