# AN ISOMORPHIC VERSION OF THE BUSEMANN-PETTY PROBLEM FOR GAUSSIAN MEASURE

A. ZVAVITCH

ABSTRACT. In this paper we provide upper and lower bounds for the Gaussian Measure of a hyperplane section of a convex symmetric body. We use those estimates to give a partial answer to an isomorphic version of the Gaussian Busemann-Petty problem.

## 1. INTRODUCTION

The standard Gaussian measure on $\mathbb{R}^n$ is given by its density:

$$\gamma_n(K) = \frac{1}{(\sqrt{2\pi})^n} \int_K e^{-\frac{|x|^2}{2}} dx,$$

where $|x|^2 = \sum_{i=1}^n |x_i|^2$.
Consider two convex symmetric bodies (convex symmetric sets with nonempty interior) $K, L \subset \mathbb{R}^n$ such that

$$\gamma_{n-1}(K \cap \xi^\perp) \le \gamma_{n-1}(L \cap \xi^\perp), \;\; \forall \xi \in S^{n-1}, \tag{1}$$

where $K \cap \xi^\perp$ denotes the section of $K$ by the hyperplane orthogonal to $\xi$. Does it follow that

$$\gamma_n(K) \le \gamma_n(L)?$$

This is a Gaussian analog of the Busemann-Petty problem (see [K] for details about the BP problem). It was shown in [Z], that the answer to the above question is affirmative if $n \le 4$ and it is negative if $n \ge 5$.

When $n \ge 5$, it is natural to ask an isomorphic version of the Gaussian Busemann-Petty problem: does there exist an absolute constant $c$ so that if $K$ and $L$ are convex symmetric bodies satisfying (1) then $\gamma_n(K) \le c\gamma_n(L)$ ?

In this paper we give a partial answer to this question:

**Theorem.** *For any $n \in \mathbb{N}$ and convex symmetric bodies $K, L \subset \mathbb{R}^n$, if*

$$\gamma_{n-1}(K \cap \xi^\perp) \le \gamma_{n-1}(L \cap \xi^\perp), \;\; \forall \xi \in S^{n-1},$$

*then*

$$\gamma_n(K) \le c\left(\gamma_n(L)\right)^{\frac{n-1}{n}}, \tag{2}$$

*where c is an absolute constant.*

This theorem will follow from two results proved below, the first one is that for any convex symmetric body $K$

$$\gamma_n(K) \leq \gamma_{n-1}(K \cap \xi^\perp), \quad \forall \xi \in S^{n-1},$$

and the second one is that for any convex symmetric body $L$ there exists a direction $\xi \in S^{n-1}$, such that

$$\gamma_{n-1}(L \cap \xi^\perp) \leq c\,(\gamma_n(L))^{\frac{n-1}{n}}. \tag{3}$$

It would be interesting to try to remove the power $(n-1)/n$ in the right hand side of (2), since it will then solve the same problem for Lebesgue measure (the so-called slicing problem, see [MP1] for more details). We must note that (3) is sharp (consider a sequence of Euclidean balls with radii going to 0), so the theorem can not be improved using the method described below.

**Acknowledgment.** The author is grateful to Alexander Koldobsky, Mark Rudelson and the anonymous referee for many suggestions that improved the paper.

## 2. Proof of the theorem

It was proved in [BS], that if $K \subset \mathbb{R}^n$ is a convex closed set with $\gamma_n(K) \geq \frac{1}{2}$, then $\gamma_{n-1}(K \cap \xi^\perp) \geq \frac{1}{2}$ for all $\xi \in S^{n-1}$. The next simple lemma shows that this result can be improved in the case of $K$ being symmetric.

**Lemma 1.** *For any convex symmetric body $K \in \mathbb{R}^n$:*

$$\gamma_n(K) \leq \gamma_{n-1}(K \cap \xi^\perp), \quad \forall \xi \in S^{n-1}.$$

**Proof :** By rotation invariance of the Gaussian measure, it is enough to consider the case $\xi = (0, \dots, 0, 1)$. Put

$$K(t) = K \cap ((0, \cdots, 0, 1)^\perp + t(0, \cdots, 0, 1))$$

and

$$PK(t) = K(t)\big|\mathbb{R}^{n-1} = \text{ orthogonal projection of } K(t) \text{ to } \mathbb{R}^{n-1}.$$

Note that $K$ is a convex body, so for $0 < \lambda < 1$

$$\lambda K(a) + (1-\lambda)K(b) \subset K(\lambda a + (1-\lambda)b),$$

thus

$$\lambda PK(a) + (1-\lambda)PK(b) \subset PK(\lambda a + (1-\lambda)b).$$

In particular,

$$\frac{1}{2}PK(t) + \frac{1}{2}PK(-t) \subset PK(0) = K(0). \tag{4}$$

Next use that $\gamma_{n-1}$ is a log-concave measure ([Bo], [E] or [Li]):

$$\log \gamma_{n-1}(\lambda A + (1-\lambda)B) \geq \lambda \log \gamma_{n-1}(A) + (1-\lambda)\log \gamma_{n-1}(B).$$

The latter inequality together with (4) gives

$$\log \gamma_{n-1}(K(0)) \geq \log \gamma_{n-1}(\frac{1}{2}PK(t) + \frac{1}{2}PK(-t))$$

$$\geq \frac{1}{2}\log \gamma_{n-1}(PK(t)) + \frac{1}{2}\log \gamma_{n-1}(PK(-t)) = \log \gamma_{n-1}(PK(t)).$$

So $\gamma_{n-1}(K(0)) \geq \gamma_{n-1}(PK(t))$.

Now we are ready to finish the proof:

$$\gamma_n(K) = \frac{1}{(\sqrt{2\pi})^n}\int_K e^{-\frac{|x|^2}{2}}dx = \frac{1}{(\sqrt{2\pi})^n}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\int_{K(t)} e^{-\frac{\sum_{i=1}^{n-1} x_i^2}{2}}dxdt$$

$$= \frac{1}{(\sqrt{2\pi})^n}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\int_{PK(t)} e^{-\frac{\sum_{i=1}^{n-1} x_i^2}{2}}dxdt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\gamma_{n-1}(PK(t))dt$$

$$\leq \gamma_{n-1}(K(0))\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}dt = \gamma_{n-1}(K(0)). \quad (5)$$

□

**Remark 1:** It is clear that (5) actually shows:

$$\gamma_n(K) \leq \gamma_{n-1}(K\cap\xi^{\perp})\gamma_1(K\big|\xi), \quad \forall \xi \in S^{n-1}, \quad (6)$$

where $K\big|\xi$ is the orthogonal projection of $K$ onto $\xi$.

Moreover it was pointed to us by the referee, that (6) can be generalized to

$$\gamma_n(K) \leq \gamma_{n-k}(K\cap E^{\perp})\gamma_k(K\big|E), \quad (7)$$

for any $k$-dimensional subspace $E$ of $\mathbb{R}^n$. Actually (7) is true for a general convex body $K$ whose Gaussian-barycenter is 0: $\int_K x d\gamma_n(x) = 0$. This can be shown by straightforward generalization of a result of Milman and Pajor (see [MP2], Remark 2 on page 319 and Lemma 1 on page 316).

Our next goal is to find an upper bound for the Gaussian measure of a hyperplane section. First we will consider the case where $K$ is a regular Euclidean ball of radius $t > 0$.

**Lemma 2.** *For any $t > 0$ and $n \in \mathbb{N}$,*

$$c_1\left(\gamma_n(tB_2^n)\right)^{\frac{n-1}{n}} \leq \gamma_{n-1}(tB_2^{n-1}) \leq c_2\left(\gamma_n(tB_2^n)\right)^{\frac{n-1}{n}}, \quad (8)$$

*where $tB_2^n = \{x \in \mathbb{R}^n : |x| \leq t\}$ and $c_1, c_2$ are absolute constants.*

**Proof :** First note that for any $t \geq 0$

$$\int_0^t r^{n-2} e^{-\frac{r^2}{2}} dr \leq \frac{n^{\frac{n-1}{n}}}{n-1} \left( \int_0^t r^{n-1} e^{-\frac{r^2}{2}} dr \right)^{\frac{n-1}{n}}. \tag{9}$$

To prove this inequality consider a function

$$f(t) = \int_0^t r^{n-2} e^{-\frac{r^2}{2}} dr - \frac{n^{\frac{n-1}{n}}}{n-1} \left( \int_0^t r^{n-1} e^{-\frac{r^2}{2}} dr \right)^{\frac{n-1}{n}},$$

then

$$\begin{aligned} f'(t) &= t^{n-2} e^{-\frac{t^2}{2}} - n^{-\frac{1}{n}} t^{n-1} e^{-\frac{t^2}{2}} \left( \int_0^t r^{n-1} e^{-\frac{r^2}{2}} dr \right)^{-\frac{1}{n}} \\ &= t^{n-2} e^{-\frac{t^2}{2}} \left( 1 - n^{-\frac{1}{n}} t \left( \int_0^t r^{n-1} e^{-\frac{r^2}{2}} dr \right)^{-\frac{1}{n}} \right) \\ &< t^{n-2} e^{-\frac{t^2}{2}} \left( 1 - n^{-\frac{1}{n}} t \left( \int_0^t r^{n-1} dr \right)^{-\frac{1}{n}} \right) = 0. \end{aligned}$$

Thus $f(t)$ is a decreasing function and $f(t) \leq 0$ for $t > 0$, which proves the inequality (9).

Passing to polar coordinates and applying (9) we get

$$\begin{aligned} \gamma_{n-1}(tB_2^{n-1}) &= \frac{1}{(\sqrt{2\pi})^{n-1}} \int_{S^{n-2}} \int_0^t r^{n-2} e^{-\frac{r^2}{2}} dr d\theta \\ &= \frac{|S^{n-2}|}{(\sqrt{2\pi})^{n-1}} \int_0^t r^{n-2} e^{-\frac{r^2}{2}} dr \\ &\leq \frac{|S^{n-2}| n^{\frac{n-1}{n}}}{(\sqrt{2\pi})^{n-1}(n-1)} \left( \int_0^t r^{n-1} e^{-\frac{r^2}{2}} dr \right)^{\frac{n-1}{n}} \\ &= \frac{|S^{n-2}| n^{\frac{n-1}{n}}}{|S^{n-1}|^{\frac{n-1}{n}}(n-1)} \left( \gamma_n(tB_2^n) \right)^{\frac{n-1}{n}}. \end{aligned} \tag{10}$$

Finally, from $|S^{n-1}| = 2\pi^{n/2}/\Gamma(\frac{n}{2})$ and the Stirling asymptotic formula for the Gamma-function we get

$$\gamma_{n-1}(tB_2^{n-1}) \leq c_2 \left( \gamma_n(tB_2^n) \right)^{\frac{n-1}{n}}.$$

To prove the lower bound for $\gamma_{n-1}(tB_2^{n-1})$. We first note that if $t \geq 2\sqrt{n}$ then $\gamma_{n-1}(tB_2^{n-1}) > c$ (see [Li] or [Ba], p. 236), thus

$$\gamma_{n-1}(tB_2^{n-1}) > c \left( \gamma_n(tB_2^n) \right)^{\frac{n-1}{n}}.$$

If $t < 2\sqrt{n}$ it is easy to show that

$$\left(\int_0^t r^{n-1} e^{-\frac{r^2}{2}} dr\right)^{\frac{n-1}{n}} < C \int_0^t r^{n-2} e^{-\frac{r^2}{2}} dr. \tag{11}$$

Indeed

$$\left(\int_0^t r^{n-1} e^{-\frac{r^2}{2}} dr\right)^{-\frac{1}{n}} \le \left(\int_0^t r^{n-1} e^{-2n} dr\right)^{-\frac{1}{n}} \le Ct^{-1}$$

and

$$\left(\int_0^t r^{n-1} e^{-\frac{r^2}{2}} dr\right)^{\frac{n-1}{n}} < C \int_0^t r^{n-1} t^{-1} e^{-\frac{r^2}{2}} dr < C \int_0^t r^{n-2} e^{-\frac{r^2}{2}} dr.$$

Finally, the lemma follows from (11) and computations similar to (10).

□

To prove the next lemma we first remind the definition of an intersection body. Let $K$ and $M$ be origin symmetric star bodies in $\mathbb{R}^n$. Following Lutwak [Lu], we say that $K$ is the *intersection body of* $M$ if the radius of $K$ in every direction is equal to the volume of the central hyperplane section of $M$ perpendicular to this direction:

$$\|\xi\|_K^{-1} = \mathrm{Vol}_{n-1}(M \cap \xi^\perp), \ \ \forall \xi \in S^{n-1}.$$

A more general class of *intersection bodies* is defined as the closure in the radial metric of the class of intersection bodies of star bodies. It is clear that $tB_2^n$ is an intersection body (see [Lu], [K] for more results and examples of intersection bodies). It was proved in [Z] (Theorems 3, 4), that if $K$ is intersection body then the answer to the Gaussian Busemann-Petty problem is affirmative for $K$ and any convex symmetric body $L$, whose central sections have greater Gaussian measure than the corresponding sections of $K$.

**Lemma 3.** *For any convex symmetric body $K \subset \mathbb{R}^n$, there exists a direction $\xi \in S^{n-1}$ such that*

$$\gamma_{n-1}(K \cap \xi^\perp) \le c\,(\gamma_n(K))^{\frac{n-1}{n}},$$

*where $c$ is an absolute constant.*

**Proof :** Assume that for any $\xi \in S^{n-1}$

$$\gamma_{n-1}(K \cap \xi^\perp) > c_2\,(\gamma_n(K))^{\frac{n-1}{n}},$$

where $c_2$ is the absolute constant from Lemma 1. Consider $t > 0$ so that

$$\gamma_{n-1}(tB_2^{n-1}) = c_2\,(\gamma_n(K))^{\frac{n-1}{n}}. \tag{12}$$

Then

$$\gamma_{n-1}(K\cap\xi^{\perp}) > \gamma_{n-1}(tB_2^n\cap\xi^{\perp}),\ \ \forall\xi\in S^{n-1}. \tag{13}$$

But $tB_2^n$ is an intersection body, therefore (13) implies

$$\gamma_n(K) > \gamma_n(tB_2^n).$$

The latter inequality together with (12) gives:

$$(c_2^{-1}\gamma_{n-1}(tB_2^{n-1}))^{\frac{n}{n-1}} > \gamma_n(tB^n)$$

or

$$\gamma_{n-1}(tB_2^{n-1}) > c_2\left(\gamma_n(tB_2^n)\right)^{\frac{n-1}{n}},$$

which contradicts Lemma 2.

□

**Remark 2:** Using a similar method, one can show that for any intersection body $K\subset\mathbb{R}^n$, there exists a direction $\xi \in S^{n-1}$ such that

$$c_1\left(\gamma_n(K)\right)^{\frac{n-1}{n}} < \gamma_{n-1}(K\cap\xi^{\perp}),$$

where $c_1$ is the absolute constant from Lemma 1. Indeed, assuming the latter inequality not to be true we get

$$\gamma_{n-1}(K\cap\xi^{\perp}) < \gamma_{n-1}(tB_2^n\cap\xi^{\perp}),\ \ \forall\xi\in S^{n-1},$$

where $t$ is such that $\gamma_{n-1}(tB_2^{n-1}) = c_1\left(\gamma_n(K)\right)^{\frac{n-1}{n}}$. Again Gaussian Busemann-Petty problem has an affirmative answer in this case, thus

$$c_1\left(\gamma_n(tB_2^n)\right)^{\frac{n-1}{n}} > \gamma_{n-1}(tB_2^{n-1}),$$

which again contradicts Lemma 2.

**Proof of the Theorem:** From Lemma 3 we get that there is a $\xi \in S^{n-1}$ such that

$$\gamma_{n-1}(L\cap\xi^{\perp}) \le c\left(\gamma_n(L)\right)^{\frac{n-1}{n}},$$

but then, from Lemma 1:

$$\gamma_n(K) \le \gamma_{n-1}(K\cap\xi^{\perp}).$$

Using

$$\gamma_{n-1}(K\cap\xi^{\perp}) \le \gamma_{n-1}(L\cap\xi^{\perp}),\ \ \ \forall\xi\in S^{n-1},$$

we get

$$\gamma_n(K) \le c\left(\gamma_n(L)\right)^{\frac{n-1}{n}}.$$

□

## References


[BS] W. Banaszczyk, S. J. Szarek, *Lattice coverings and Gaussian measures of n-dimensional convex bodies,* Discrete Comput. Geom. **17** (1997), no. 3, 283–286.

[Ba] A. Barvinok, *A course in convexity* Graduate Studies in Mathematics, 54. American Mathematical Society, Providence, RI, (2002).

[Bo] C. Borell, *The Brunn-Minkowski inequality in Gauss space,* Invent. Math. **30** (1975), no. 2, 207-216.

[E] A. Ehrhard, *Symetrisation dans l'espace de Gauss [Symmetrization in Gaussian spaces] (in French)* Math. Scand. **53** (1983), no. 2, 281–301.

[K] A. Koldobsky, *Sections of star bodies and the Fourier transform*, Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), 225-247, Contemp. Math., **320**, Amer. Math. Soc., Providence, RI, 2003.

[Li] M. A. Lifshits, *Gaussian random functions Mathematics and its Applications*, 322. Kluwer Academic Publishers, Dordrecht, 1995.

[Lu] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232-261.

[MP1] V. D. Milman, A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space,* Geometric aspects of functional analysis (1987–88), 64-104, Lecture Notes in Math., **1376**, Springer, Berlin, 1989.

[MP2] V. D. Milman, A. Pajor, *Entropy and asymptotic geometry of non-symmetric convex bodies*, Adv. Math. **152** (2000), no. 2, 314-335.

[Z] A. Zvavitch, *Gaussian Measure of Sections of convex bodies*, Adv. Math., to appear.



A. Zvavitch, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

*E-mail address*: zvavitch@math.missouri.edu