Theory of Numbers (47011/57011)
Take Home Exam 2
Instructor: Prof. Artem Zvavitch
You may use any books, notes... BUT, PLEASE, DO IT YOURSELF, I may and will ask you to explain to me a solution which I may find suspicious, if you would not be able to explain what you wrote the grade for the exam will be "F". Exam, MUST be submitted before 11 a.m. TUESDAY, APRIL 7, 2015. If you can not make for the class this day - put it in my mailbox in advance. If you can not come to university scan and email it to me. Exam is NOT simple. Please, Please, start it now.

Problem 1. *Find (without using calculator!)*

$$10^{35} \bmod 7.$$

Problem 2. *Let $a = a_r a_{r-1} \dots a_1 a_0$ be the decimal representation of a. Please prove that*

- $7|a$ *iff* $7|a_r \dots a_1 - 2a_0$ *(Example:* $7|105$ *because* $10 - 2*5 = 0$ *and* $7|0$*, but* $7 \not| 356$ *because* $35 - 2*6 = 23$ *and* $7 \not| 23$*).*
- $13|a$ *iff* $13|a_r \dots a_1 - 9a_0$ *(Example:* $13|299$ *because* $29 - 9*9 = -52$ *and* $13|52$*, but* $13 \not| 356$ *because* $35 - 9*6 = -19$ *and* $13 \not| -19$*).*

Problem 3. *Consider $n \geq 2$. Prove that if a and n are relatively prime, then there exists a unique, modulo n, a^* such that*

$$aa^* \equiv 1 (\text{mod } n).$$

We call a^* the inverse of a **modulo** *n.*

Problem 4. *Show that the inverse of 2 modulo 7 is not the inverse of 2 modulo 15.*

Problem 5. *Find a solution of the system of congruences:*

$$\begin{cases} 11x + 5y = 7(mod\ 20) \\ 6x + 3y = 8(mod\ 20) \end{cases}$$

Problem 6. *Consider a prime $p > 2$. Prove that $a^{2p-1} \equiv a$ (mod $2p$) for any integer a.*

Problem 7. *Consider p and q are distinct primes, prove that for any integer a*

$$pq | a^{pq} - a^p - a^q + a.$$

Problem 8. *Let $\sigma^*(n) = \sigma(n) - n$, i.e. $\sigma^*(n)$ is the sum of all proper devisors of n (a proper divisor of n is a divisor which is strictly less then n). Show that $\sigma^*(n)$ is not a multiplicative function.*

Problem 9. *Show that if n is a composite number then $\sigma(n) > n + \sqrt{n}$.*

Problem 10. *Prove that*

$$\sum_{d|n} |\mu(d)| = 2^{\omega(n)},$$

where $\omega(n)$ is the number of distinct prime divisors of n (you may start by proving that the left hand side is a multiplicative function of n).