Functions of Real Variables 1 (62051/72051)
Home Work 10, due on Monday November 28.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Prove the following variant of the* **Vitali covering lemma:** *If E is covered in the Vitali sense by a family $\mathcal{B}$ of balls, and $m_*(E) \in (0, \infty)$, then for every $\nu > 0$ there exists a disjoint collection of balls $\{B_j\}_{j=1}^{\infty}$ in $\mathcal{B}$ such that*

$$m_*(E \setminus \bigcup_{j=1}^{\infty} B_j) = 0 \text{ and } \sum_{j=1}^{\infty} |B_j| \leq (1+\nu) m_*(E).$$

Problem 2. *Show that the case of equality in the Cauchy-Schwarz inequality (i.e. $|(f,g)| = \|f\| \|g\|$) is possible if and only if $f = cg$ for some $c \in \mathbb{R}$.*

Problem 3. *Use the definition of $\ell_2(\mathbb{Z})$ to prove that this space is complete and separable.*

Problem 4. *Give an example of a function $f \in L_2(\mathbb{R}^d)$ but $f \notin L_1(\mathbb{R}^d)$. Also give an example of a function $g \in L^1(\mathbb{R}^d)$ but $g \notin L_2(\mathbb{R}^d)$. Also prove that if $f(x)$ is a bounded function in $L_1(\mathbb{R}^d)$ then $f \in L_2(\mathbb{R}^d)$.*

Problem 5. *Prove that simple functions and continuous functions of compact support are dense subspaces of $L_2(\mathbb{R}^d)$.*