Functions of Real Variables 1 (62051/72051)
Home Work 11, due on Friday, December 9.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Suppose* $\{g_k\}_{k=1}^{\infty}$ *is an orthonormal basis for* $L^2(\mathbb{R}^d)$*. Prove that the collection* $\{g_{k,j}\}_{1\le j,k\le\infty}$*, where*

$$g_{k,j}(x,y) = g_k(x)g_j(y)$$

is an orthonormal basis for $L_2(\mathbb{R}^{2d}$. **Suggesions:** *Note that* $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$ *and use the Fubini theorem to verify the orthonormality. Next consider the case* $(f, g_{k,j}) = 0$ *for all* k, j *and again use Fubini theorem.*

Problem 2. *Let* $\nu(t)$ *be a fixed, continuous strictly positive function on* $[a, b]$*. Define* $L_2([a, b], \nu)$ *to be the space of all measurable functions* f *on* $[a, b]$ *such that*

$$\int_a^b |f(t)|^2 \nu(t)dt < \infty.$$

Define the inner product on $L_2([a, b], \nu)$ *as*

$$(f(t), g(t))_\nu = \int_a^b f(t)\overline{g(t)}\nu(t)dt.$$

Prove that $L_2([a, b], \nu)$ *is a Hilbert space and find the mapping which gives a unitary correspondence between* $L_2([a, b], \nu)$ *and* $L_2([a, b])$*.*

Problem 3. *Let* S *be a subspace of a Hilbert space* H*. Prove that* $(S^\perp)^\perp$ *is the smallest closed subspace of* H *that contains* S*.*

Problem 4. *Let* P *be the orthogonal projection associated with a closed subspace* S *in a Hilbert space* H*, that is* P *is a linear operator such that*

$$P(f) = f \text{ if } f \in S \text{ and } P(f) = 0 \text{ if } f \in S^\perp.$$

- *Show that* $P^2 = P$ *and* $P^* = P$*.*
- *Conversely, if* P *is any bounded operator satisfying* $P^2 = P$ *and* $P^* = P$*, prove that* P *is the orthogonal projection for some closed subspace of* H*.*
- *Using* P *prove that if* S *is a closed subspace of a separable Hilbert space, then* S *is also a separable Hilbert space.*