

Functions of Real Variables 1 (62051/72051)
Home Work 5, due on Wednesday October 5.
Instructor: Prof. Artem Zvavitch.

Problem 1. (The Borel-Cantelli lemma): Suppose $\{E_k\}_{k=1}^{\infty}$ is countable family of measurable subsets of \mathbb{R}^d and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$E = \{x \in \mathbb{R}^d : x \in E_k, \text{ for infinitely many } k\} = \limsup_{k \rightarrow \infty} (E_k).$$

- Show that E is measurable.
- Prove that $m(E) = 0$.

Hint: write $E = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k$.

Problem 2. Let $\{f_n\}$ be a sequence of measurable functions on $[0, 1]$ with $|f_n(x)| < \infty$ for a.e. x . Show that there exists a sequence c_n of positive real numbers such that

$$\frac{f_n(x)}{c_n} \rightarrow 0, \text{ a.e. } x.$$

Hint: Pick c_n such that $m(\{x : |f_n(x)/c_n| > 1/n\}) < 2^{-n}$, and apply the Borel-Cantelli lemma.

Problem 3. Prove that there is a continuous function that maps a Lebesgue measurable set to non-measurable set. Discussion around Problem 3 in HW 4 may help.

Problem 4. Show that there is no function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

- f is continuous.
- $f = \chi_{[0,1]}$ a.e. on \mathbb{R} .