

**Functions of Real Variables 1 (62051/72051)**  
**Home Work 7, due on Monday October 24.**  
**Instructor: Prof. Artem Zvavitch.**

**Problem 1. Volume of the Unit Ball:** We proved before that if  $B = \{x \in \mathbb{R}^d : |x| \leq 1\}$ , then  $m(rB) = r^d m(B)$  for all  $r \geq 0$ . Now it is the time to provide the formula for  $v_d = m(B)$ :

- Use Fubini's theorem (Corollary of Fubini's theorem) to show that

$$v_2 = 2 \int_{-1}^1 (1 - x^2)^{1/2} dx,$$

and show (using standard tricks for Riemann integral) that  $v_2 = \pi$ .

- Use similar methods to show that

$$v_d = 2v_{d-1} \int_0^1 (1 - x^2)^{(d-1)/2} dx.$$

- Show that

$$v_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}.$$

Where  $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ .

- Prove (use induction) that  $\Gamma(n) = (n - 1)!$  and write a formula for  $v_{2k+1}$ .

**Problem 2.** Suppose  $f \in L_1(\mathbb{R}^d)$ . For each  $t > 0$  let  $E_t = \{x : |f(x)| > t\}$ . Prove that

$$\int_{\mathbb{R}^d} |f(x)| dx = \int_0^\infty m(E_t) dt.$$

**Problem 3. Convolution of two functions.** Consider two measurable functions  $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$ .

- Prove that  $f(x - y)g(y)$  is measurable of  $\mathbb{R}^{2d}$ .
- Show that if  $f, g \in L_1(\mathbb{R}^d)$ , then  $f(x - y)g(y) \in L_1(\mathbb{R}^{2d})$ .
- Now we can define a convolution operator, which takes two functions  $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$  and gives function  $f * g : \mathbb{R}^d \rightarrow \mathbb{R}$  defined as

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y) dy.$$

Show that  $f * g$  is well defined for a.e.  $x$  (i.e.  $f(x - y)g(y)$  is integrable for a.e.  $x$ ).

- Show that  $f * g$  is integrable whenever  $f$  and  $g$  are integrable, and that

$$\|f * g\|_{L_1(\mathbb{R}^d)} \leq \|f\|_{L_1(\mathbb{R}^d)} \|g\|_{L_1(\mathbb{R}^d)}.$$

- Show that  $f * g$  is uniformly continuous when  $f$  is integrable and  $g$  bounded.
- Show that if in addition to above  $g$  is integrable, then

$$(f * g)(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$