Functions of Real Variables 1 (62051/72051)
Home Work 8, due on Monday October 31.
Instructor: Prof. Artem Zvavitch.
I WILL ADD MORE PROBLEMS ON WEDNESDAY

Problem 1. *Prove that if fF is integrable on $\mathbb{R}^d$, and f is not identically zero, them*

$$f^*(x) \geq \frac{c}{|x|^d}, \text{ for some } c > 0 \text{ and all } |x| \geq 1.$$

Conclude that f^* is not integrable on $\mathbb{R}^d$. Then, show that the weak type estimate

$$m(\{x : f^*(x) > \alpha\}) \leq \frac{c}{\alpha}$$

for all $\alpha > 0$, whenever $\int |f| = 1$ is best possible in the following sense: if f is supported in the unit ball with $\int |f| = 1$, then

$$m(\{x : f^*(x) > \alpha\}) \geq \frac{c'}{\alpha}$$

for some $c' > 0$ ad all sufficiently small α.

I would suggest to start with an observation that if f is not identically zero then there always exists a ball B such that $\int_B |f| > 0$.

Problem 2. *Consider the function on $\mathbb{R}$ defined by*

$$f(x) = \begin{cases} \frac{1}{|x|(\log 1/|x|)^2}, & \text{if } |x| \leq 1/2 \\ 0, & \text{otherwise} \end{cases}$$

- *Show that f is integrable.*
- *Establish the inequality*

$$f^*(x) \geq \frac{c}{|x|(\log 1/|x|)} \text{ for some } c > 0 \text{ and } |x| \leq 1/2,$$

  to conclude that the maximal function f^* is not locally integrable.