Functions of Real Variables 1 (62051/72051)
Home Work 9, due on Monday November 14.
Instructor: Prof. Artem Zvavitch.

Problem 1. *Suppose* $\{K_\delta\}$ *is a family of kernels on* $\mathbb{R}^d$ *that satisfy*

(1) $|K_\delta(x)| \le A\delta^{-d}$ *for all* $\delta > 0$,
(2) $|K_\delta(x)| \le A\delta/|x|^{d+1}$ *for all* $\delta > 0$,
(3) $\int_{\mathbb{R}^d} K_\delta(x)dx = 0$ *for all* $\delta > 0$.

Show that if $f \in L_1(\mathbb{R}^d)$, *then*

$$(f * K_\delta)(x) \to 0 \text{ for a.e. } x, \text{ as } \delta \to 0.$$

Problem 2. *Prove that if a measurable subset* $E \subset [0,1]$ *satisfies* $m(E \cap I) \ge \alpha m(I)$, *for some* $\alpha > 0$ *and all intervals* $I \subset [0,1]$, *then* $m(E) = 1$. **Hint:** *A corollary about points of density proved in the class, may help.*

Problem 3. *Assume that* F *is continuous on* $[a,b]$. *Show that*

$$D^+(F)(x) = \limsup_{h \to 0^+} \frac{F(x+h) - F(x)}{h}$$

is measurable.

Problem 4. *Assume* F *is of bounded variation and continuous. Prove that* $F = F_1 - F_2$, *where both* F_1 *and* F_2 *are monotonic and continuous.*

Problem 5. *Show that if* F *is of bounded variation in* $[a,b]$, *then*

- $\int_a^b |F'(x)|dx \le T_F(a,b)$.
- $\int_a^b |F'(x)|dx = T_F(a,b)$ *iff* F *is absolutely continuous.*

Use the las result to show that the formula $L = \int_a^b |z'(t)|dt$ *for the length of a rectifiable curve parametrized by* $z(t)$ *holds iff* z *is absolutely continuous.*

Problem 6. *Let* $f : \mathbb{R} \to \mathbb{R}$. *Prove that* f *satisfies the Lipschitz condition*

$$|f(x) - f(y)| \le M|x - y|,$$

for some M *and all* $x, y \in \mathbb{R}$, *iff* f *satisfies the following two properties:*

- f *is absolutely continuous.*
- $|f'(x)| \le M$.