

Functions of Real Variables 1 (62051/72051)
Final Home Work , due on December 13.
Each Problem is 17 points, maximum grade is 100.
Instructor: Prof. Artem Zvavitch.

Problem 1. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of nonempty (Lebesgue) measurable subsets of $[0, 1]$ satisfying

$$\lim_{n \rightarrow \infty} m(E_n) = 1.$$

Show that for each $\varepsilon \in (0, 1)$ there exists a subsequence $\{E_{n_k}\}_{k=1}^{\infty}$ of $\{E_n\}_{n=1}^{\infty}$ such that

$$m(\cap_{k=1}^{\infty} E_{n_k}) \geq \varepsilon$$

Problem 2. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be any function (not necessarily measurable!). Prove that the set of points $x \in \mathbb{R}$ such that

$$F(y) \leq F(x) \leq F(z)$$

for all $y \leq x$ and $z \geq x$ is Borel set.

Problem 3. Suppose that $f_n : \mathbb{R}^d \rightarrow \mathbb{R}$ is a sequence of measurable function that converge to a measurable function f mM -almost everywhere. In addition, suppose that there is a non-negative measurable function F such that

$$\int_{\mathbb{R}^d} F dx < \infty \text{ and } |f_n| \leq F \text{ for all } n.$$

Prove that

$$\int_{\mathbb{R}^d} \limsup f_n dx \geq \limsup \int_{\mathbb{R}^d} dx.$$

Also, please, give an example to show that the above conclusion may fail without the assumption of the existence of the integrable dominating function F .

Problem 4. Suppose that $A \subset \mathbb{R}$ satisfies $m_1(A) = 0$, where m_1 denotes the one-dimensional Lebesgue measure. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}^2$ satisfies

$$|f(x) - f(y)| \leq \sqrt{|x - y|}, \text{ for every } x, y \in \mathbb{R}.$$

Show that $m_2(f(A)) = 0$, m_2 denotes the two-dimensional Lebesgue measure on \mathbb{R}^2 .

Problem 5. Let f be absolutely continuous in the interval $[\varepsilon, 1]$ for each $0 < \varepsilon < 1$. Does the continuity off f at 0 imply that f is absolutely continuous on $[0, 1]$? What if f is also of bounded variation on $[0, 1]$?

Problem 6. For a closed convex nonempty subset K of a Hilbert space H and $x \in H$, denote by $Px \in K$ a unique closest point to x among points in K , i.e. $Px \in K$ such that

$$\|Px - x\| \leq \|y - x\|, \text{ for all } y \in K.$$

First show that such point Px exists and unique. Next prove that all $x, y \in H$

$$\|Px - Py\| \leq \|x - y\|.$$