MATH- 62052/72052
Functions of Real Variables 2.
Lecture 19.

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Let F be an increasing function on $[a, b]$.

Brief reminder from Tuesday (03/10/2020): Lebesgue-Stieltjes integral & Borel Measure on $\mathbb{R}$

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We will consider increasing functions which are *normalized*, i.e. we will require it to be right-continuous on $[a, b]$ i.e. $F(x_0) = \lim_{x \to x_0^+} F(x)$.

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Conversely, if μ is a measure on $\mathcal{B}$ that is finite on all bounded intervals, then

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The idea of the proof: we used the structure of $\mathbb{R}$ and introduced an exterior measure μ_* on $\mathbb{R}$

$$\mu_*(E) = \inf \sum (F(b_j) - F(a_j)),$$

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where the infimum is taken over all covers of E by $\cup(a_j, b_j]$. We studied the properties of μ_* and used extension theorems.

Remarks:

- We note that "many" functions F can produce the same measure μ indeed take $G(x) = F(x) + c$, where $c \in \mathbb{R}$ is a constant. Then $G(b_i) - G(a_i) = F(b_i) - F(a_i)$ and thus generate the same exterior measure μ_*.

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- We note that any increasing normalized function F on the closed interval $[a, b]$ can be extended to an increasing normalized function F on $\mathbb{R}$:

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It is easy to see that if $E \subset \mathbb{R} \setminus [a, b]$, then $\mu_*(E) = 0$ and thus the resulting measure μ the intervals $(-\infty, a]$ and (b, ∞) has measure zero.