MATH- 62052/72052
Functions of Real Variables 2.
Lecture 27.

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Spring, 2020

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Moreover, from above $d(f,g) = \|f-g\|_1$ is a metric on L^1, and we were able to show that L^1 is complete (i.e. Cauchy sequence is convergent). We were also able to show that $L^1(\mathbb{R}^d)$ is separable, i.e. there is a countable collection of $\{f_k\}_{k=1}^{\infty}$ such that their linear combination is dense in $L^1(\mathbb{R}^d)$ (we can show that $L^1(X)$ is separable, but here we need to ask X to be separable).

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Actually to prove the triangle inequality (2) we used Cauchy-Schwartz inequality:

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Moreover, from above $d(f,g) = \|f-g\|_2$ is a metric on L^2, and we were able to show that L^2 is complete. We were also able to show that $L^2(\mathbb{R}^d)$ is separable (we can show that $L^2(X)$ is separable, but here we need to ask X to be separable).

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Note, there are $L^p(X, \mathcal{M}, \mu)$, $p \in (0, 1)$, those are not a normed spaces ($\| \cdot \|_p$ does not satisfy triangle inequality if $0 < p < 1$). There is also a very interesting notion for L^p when $p \leq 0$.

We can consider $L^p(\mathbb{R}^d, \mathcal{M}, m)$ consists of all complex (real) Lebesgue measurable functions on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |f(x)|^p dx < \infty$$

With the norm

$$\|f\|_p = \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

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Again we also have a metric on ℓ^p defined as

$$\|a-b\|_p = \left(\sum_{n=1}^{\infty} |a_n - b_n|^p\right)^{\frac{1}{p}},$$

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We can also consider a finite dimensional ℓ_d^p, one can see it as $X = \{1, 2, \dots, d\}$ and μ is a counting measure. Then a measurable function on X is simply a sequence $a = \{a_n\}_{n=1}^{d}$. We define the space ℓ_d^p to be the space of all complex (real) sequences such that $\|a\|_p = \left(\sum_{n=1}^{d} |a_n|^p \right)^{1/p}$.

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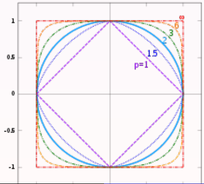

Towards "triangle inequality": Hölder Inequality.

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This inequality can be generalized to $\lambda a + (1-\lambda) b \geq a^\lambda b^{1-\lambda}$ for all $a, b \geq 0$ and $\lambda \in [0, 1]$.

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$$\frac{1}{p} + \frac{1}{q} = 1 \text{ or } q = \frac{p}{p-1}.$$

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$$\frac{a+b}{2} \geq \sqrt{ab}, \text{ for all } a, b \geq 0.$$

This inequality can be generalized to $\lambda a + (1-\lambda)b \geq a^{\lambda} b^{1-\lambda}$ for all $a, b \geq 0$ and $\lambda \in [0, 1]$. One of the ways to see this inequality is to use concavity of the log function.

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Plug back $\tilde{f}(x) = f(x)/\|f\|_p$ and $\tilde{g} = g(x)/\|g\|_q$ to finish the proof and get $\|fg\|_1 \le \|f\|_p\|g\|_q$.

□

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$$\begin{aligned}
\|f+g\|_p^p &= \int_X |f(x)+g(x)|^p d\mu \\
&\le \int_X |f(x)||f(x)+g(x)|^{p-1} d\mu + \int_X |g(x)||f(x)+g(x)|^{p-1} d\mu \\
&\le \|f\|_p \left(\int_X (|f(x)+g(x)|^{p-1})^q d\mu\right)^{1/q} + \|g\|_p \left(\int_X (|f(x)+g(x)|^{p-1})^q d\mu\right)^{1/q} \\
&\le \left(\left(\int_X |f(x)+g(x)|^p d\mu\right)^{1/p}\right)^{p/q} (\|f\|_p + \|g\|_p) \\
&= \|f+g\|_p^{p-1}(\|f\|_p + \|g\|_p).
\end{aligned}$$

How to compare L^{p_0} and L^{p_1}.

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$$\|f\|_{p_0}^{p_0} = \int_X |f(x)|^{p_0} d\mu = \int_X |f(x)|^{p_0} \cdot 1 d\mu \leq \left(\int_X |f(x)|^{p_1} d\mu \right)^{\frac{p_0}{p_1}} \cdot \left(\int_X 1^{\frac{p_1}{p_1-p_0}} d\mu \right)^{\frac{p_1-p_0}{p_1}},$$

where the last inequality follows from Hölder inequality with $p = \frac{p_1}{p_0} > 1$ and $q = \frac{p_1}{p_1-p_0}$

□

How to compare ℓ^{p_0} and ℓ^{p_1}.

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$$\|a\|_{p_1} \le \|a\|_{p_0}$$

Proof: There is no much to prove if $p_0 = p_1$, so assume $p_1 > p_0$.

$$\|a\|_{p_1}^{p_1} = \sum_{n=1}^{\infty} |a_n|^{p_1} = \sum_{n=1}^{\infty} |a_n|^{p_0} |a_n|^{p_1 - p_0} \le \sup_n |a_n|^{p_1 - p_0} \sum_{n=1}^{\infty} |a_n|^{p_0} \le \left(\sum_{n=1}^{\infty} |a_n|^{p_0} \right)^{\frac{p_1 - p_0}{p_0}} \|a\|_{p_0}^{p_0}$$

$\square$

We can also note that if $a \in \mathbb{R}^d$ then $\|a\|_{p_1} \le \|a\|_{p_0} \le d^{\frac{p_1 - p_0}{p_0 p_1}} \|a\|_{p_1}$ for all $1 \le p_0 \le p_1 < \infty$,

$\ell_{p_0} \subset \ell_{p_1}$ if $1 \le p_0 \le p_1 < \infty$ and
$$\|a\|_{p_1} \le \|a\|_{p_0}$$

Proof: There is no much to prove if $p_0 = p_1$, so assume $p_1 > p_0$.

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□

We can also note that if $a \in \mathbb{R}^d$ then $\|a\|_{p_1} \le \|a\|_{p_0} \le d^{\frac{p_1 - p_0}{p_0 p_1}} \|a\|_{p_1}$ for all $1 \le p_0 \le p_1 < \infty$, indeed

$$\|a\|_{p_0}^{p_0} = \sum_{n=1}^{d} |a_n|^{p_0} \cdot 1 \le \left(\sum_{n=1}^{d} |a_n|^{p_1} \right)^{\frac{p_0}{p_1}} \cdot \left(\sum_{n=1}^{d} 1^{\frac{p_1}{p_1 - p_0}} \right)^{\frac{p_1 - p_0}{p_1}} = \|a\|_{p_1}^{p_0} d^{\frac{p_1 - p_0}{p_1}}$$

where we used Hölder inequality with $p = \frac{p_1}{p_0} > 1$ and $q = \frac{p_1}{p_1 - p_0}$ on $X = 1, 2, \ldots, d$ and a counting measure μ.