

MATH- 62052/72052  
Functions of Real Variables 2.  
Lecture 28.

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The space of functions  $L^p(X, \mathcal{M}, \mu)$  consists of all complex (real) measurable functions such that

$$\int_X |f(x)|^p d\mu(x) < \infty.$$

We will use different notations  $L^p(X, \mu)$ ,  $L^p(X)$  or just  $L^p$ . We defined a norm on  $L^p$ , as

$$\|f\|_p = \|f\|_{L^p} = \|f\|_{L^p(X, \mathcal{M}, \mu)} = \left( \int_X |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}.$$

For above to be a norm, we really need  $\|f\|_{L^p(X, \mathcal{M}, \mu)} = 0$  imply  $f$  to be identically zero (which is not true from  $\|f\|_{L^p(X, \mathcal{M}, \mu)} = 0$  we only get that  $f = 0$  a.e. on  $X$ ). Thus to make a precise definition we equip  $L^p$  with equivalence relation, in which  $f = g$  if  $f = g$  a.e. on  $X$ . Now  $\|f\|_p$  satisfies the properties of norm:

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- 1  $\|af\|_p = |a| \|f\|_p$ , for all  $a \in \mathbb{C}$ .
- 2  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$
- 3  $\|f\|_p = 0$  iff  $f = 0$ .

Moreover, from above  $d(f, g) = \|f - g\|_p$  is a metric on  $L^p$ , and we will show that  $L^p$  is complete.

Consider  $p \in (1, \infty)$ , we define  $q \in (1, \infty)$  to be a conjugate/dual exponent of  $p$ :

$$\frac{1}{p} + \frac{1}{q} = 1 \text{ or } q = \frac{p}{p-1}.$$

## Hölder Inequality.

Consider  $p \in (1, \infty)$  and let  $q$  be a conjugate exponent of  $p$ . Let  $f \in L^p$ ,  $g \in L^q$ , then  $fg \in L^1$  and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

## Minkowski Inequality.

Consider  $p \in (1, \infty)$  and  $f, g \in L^p$  then  $f + g \in L^p$  and  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ .

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Thus take  $\varepsilon = 2^{-1}$  to get

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$$\exists n_1 \in \mathbb{N} : \forall m \geq n_1 : \|f_{n_1} - f_m\|_p \leq 2^{-1}.$$

Next take  $\varepsilon = 2^{-2}$  to get

$$\exists n_2 > n_1 \in \mathbb{N} : \forall m \geq n_2 : \|f_{n_2} - f_m\|_p \leq 2^{-2}.$$

Note that in particular  $\|f_{n_1} - f_{n_2}\|_p \leq 2^{-1}$ . Continue the process to build the required subsequence.

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$$\|S_K(g)\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^K \|f_{n_k} - f_{n_{k+1}}\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^K 2^{-k} \leq \|f_{n_1}\|_p + 1 \leq C.$$

Now we can use that  $S_K(g)(x)$  is a monotone sequence (this was the reason to use absolute values!) Thus it is convergent for each  $x$  (may be to  $\infty$ ) but we also can use monotone convergence theorem to get that  $\int_X |S_K(g)|^p d\mu \rightarrow \int_X |g(x)|^p d\mu$ . But we just proved that  $\int_X |S_K(g)|^p d\mu < C^p$  is uniformly bounded, thus  $\int_X |g(x)|^p d\mu < \infty$ . From here we get that  $g(x)$  is finite  $\mu$  a.e.. Using that the absolute convergence is stronger than convergence we get that  $S_K(f)(x) \rightarrow f(x)$  for  $\mu$  a.e. and from  $\int_X |f|^p d\mu \leq \int_X g^p d\mu$  we get  $f \in L^p$ .

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The proof is very similar to what we have already learned about  $L^1$  and  $L^2$ . Our goal is to show that if  $\{f_n\}_{n=1}^\infty$  is a Cauchy sequence (in  $L^p$ ) then it is convergent to a limit which belongs to  $L^p$ . As before we can always consider a subsequence of  $\{f_n\}_{n=1}^\infty$  with the property that

$\|f_{n_k} - f_{n_{k+1}}\|_p \leq 2^{-k}$ . We will show that  $\{f_{n_k}\}_{k=1}^\infty \rightarrow f(x)$  for  $\mu$  a.e.  $x$ . Consider two series


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$f(x)$  and  $g(x)$  are just a notations, we need to prove that those series are convergent! Also consider

$$S_K(f)(x) = f_{n_1}(x) + \sum_{k=1}^K f_{n_k}(x) - f_{n_{k+1}}(x) \text{ and } S_K(g)(x) = |f_{n_1}(x)| + \sum_{k=1}^K |f_{n_k}(x) - f_{n_{k+1}}(x)|$$

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Fix  $\varepsilon > 0$  and pick  $N' \in \mathbb{N}$ :  $\forall m, n \geq N' : \|f_n - f_m\|_p \leq \varepsilon$ , now pick  $N > N'$ : for all  $n_k > N$ ,  $\|f - f_{n_k}\|_p \leq \varepsilon$ , then

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$$\|fg\|_1 = \int_X |f(x)g(x)| d\mu \leq \int_X \|f\|_\infty |g(x)| d\mu \leq \|f\|_\infty \|g\|_1.$$

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Thus  $\limsup_{p \rightarrow \infty} \|f\|_p \leq \limsup_{p \rightarrow \infty} \|f\|_\infty \mu(\text{supp}(f))^{1/p} = \|f\|_\infty$ . To provide the reverse inequality we note that for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that

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Indeed if for some  $\varepsilon > 0$  we have  $\mu(\{x : |f(x)| > \|f\|_\infty - \varepsilon\}) = 0$  then  $\|f\|_\infty$  is not (essentially) the best upper bound and should be replaced by  $\|f\|_\infty - \varepsilon$ .

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$$\liminf_{p \rightarrow \infty} \|f\|_p \geq \liminf_{p \rightarrow \infty} \delta^{1/p} (\|f\|_\infty - \varepsilon) = (\|f\|_\infty - \varepsilon).$$

The above is true for arbitrary  $\varepsilon > 0$  and thus  $\liminf_{p \rightarrow \infty} \|f\|_p \geq \|f\|_\infty$ .



Another interesting example would be  $X = \mathbb{N}$ , i.e. the set of natural numbers and  $\mu$  is a counting measure. Then a measurable function on  $X$  is simply a sequence  $a = \{a_n\}_{n=1}^\infty$ . Reminder: we define the space  $\ell^p$  to be the space of all complex (real) sequences such that

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We note that  $\ell^p \subset \ell^\infty$  (indeed if  $\sum |a_i|^p$  is convergent then the summands must be bounded).

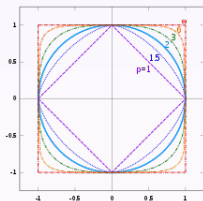
We can also consider a finite dimensional  $\ell_d^p$ , one can see it as  $X = \{1, 2, \dots, d\}$  and  $\mu$  is a counting measure. Then a measurable function on  $X$  is simply a sequence  $a = \{a_n\}_{n=1}^d$ . We define the space  $\ell_d^p$  to be the space of all complex (real) sequences such that  $\|a\|_p = \left(\sum_{n=1}^d |a_n|^p\right)^{1/p}$ . and  $\|a\|_\infty = \max |a_i|$ .





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Reminder: one can also imagine  $\ell_d^p$  as simply  $\mathbb{R}^d$  with above norm  $\|\cdot\|_p$ . Here the unit balls of  $\ell_2^p$  i.e. sets on the plane defined as  $\{x \in \mathbb{R}^2 : \|x\|_p \leq 1\}$ :



We also proved that if  $a \in \mathbb{R}^d$  then  $\|a\|_{p_1} \leq \|a\|_{p_0} \leq d^{\frac{p_1 - p_0}{p_0 p_1}} \|a\|_{p_1}$  for all  $1 \leq p_0 \leq p_1 < \infty$ . It is easy to see that the statement stays the same if  $p_1 = \infty$ . Indeed

$$\|a\|_\infty \leq \left(\sum_{n=1}^d |a_n|^{p_0}\right)^{1/p_0} \leq \|a\|_\infty \left(\sum_{n=1}^d 1\right)^{1/p_0} = d^{1/p_0} \|a\|_\infty.$$