MATH- 62052/72052
Functions of Real Variables 2.
Lecture 29.

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Normed Vector Space

Consider a vector space V over a field of scalars (real or complex numbers) together with function (to be called **norm**) $\|\cdot\| : V \to \mathbb{R}^+$ such that

1. $\|cv\| = |c|\|v\|$, for any scalar c and $v \in V$.
2. $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.
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A complete normed vector space is called a **Banach space.**

The very first examples are $\mathbb{R}$ with norm $|x|$ and $\mathbb{R}^d$ with norm $|x| = (\sum_{i=1}^{d} |x_i|^2)^{1/2}$.

We can also consider ℓ_d^p spaces, i.e. $\mathbb{R}^d$ with norm

$$\|x\|_p = \left(\sum_{n=1}^{d} |x_n|^p \right)^{1/p} \quad \text{for } p \in [1, \infty)$$

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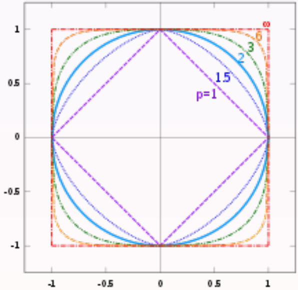
Do not forget that our standard Euclidean space is simply $\ell_d^2 = (\mathbb{R}^d, \|\cdot\|_2)$. Here the unit balls of ℓ_2^p i.e. sets on the plane defined as $\{x \in \mathbb{R}^2 : \|x\|_p \le 1\}$:



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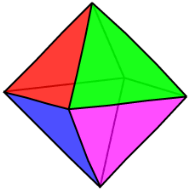
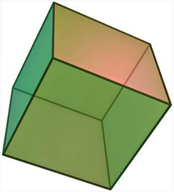
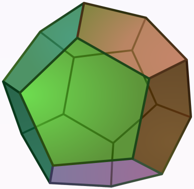

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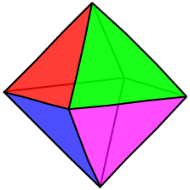
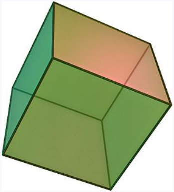
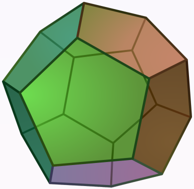

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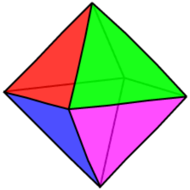
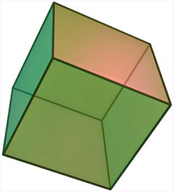
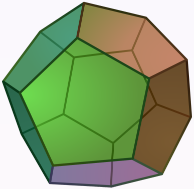

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Banach Space: Examples

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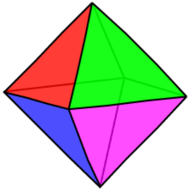
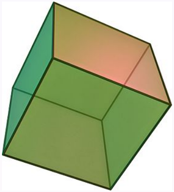
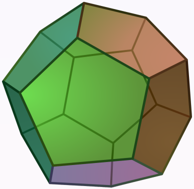

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It is also not hard to show that $(\mathbb{R}^d, \|\cdot\|_K)$ is complete and thus is a Banach space!

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space.
The vector space of functions $L^p(X, \mathcal{M}, \mu)$, $p \in [1, \infty)$ consists of all complex (real) measurable functions such that

$$\int_X |f(x)|^p d\mu(x) < \infty.$$

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Do not forget about $L^\infty(X, \mathcal{M}, \mu)$: i.e. the vector space of (essentially) bounded, measurable functions, i.e. for each f there exists a constant M: $|f(x)| < M$ for μ a.e. $x \in X$. Define

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Do not forget that $\|\cdot\|_p$ is a norm and L^p is complete with respect to is, thus L^p, $p \in [1, \infty]$ is a Banach Space.

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We can also consider the space $\Lambda(\mathbb{R})$ of bounded, Lipschitz functions, i.e. functions such that they are bounded and there exists a constant L:

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We can define a norm

$$\|f\|_{\Lambda(\mathbb{R})} = \sup_{x\in\mathbb{R}} |f(x)| + \inf_{x,y\in\mathbb{R}, x\neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Please, have fun and prove that this is indeed a Banach space!

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A linear functional is a linear mapping $\ell : \mathcal{B} \to \mathbb{R}$, i.e. a mapping such that

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$$\|\ell - \ell_n\|^* \leq 2\varepsilon,$$

or $\ell_n \to \ell$ in $\|\cdot\|^*$.

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Suppose $p \in [1, \infty)$ and $1/p + 1/q = 1$. Then

$$L_p^* = L_q.$$