

MATH- 62052/72052
Functions of Real Variables 2.
Lecture 30.

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A super brief reminder:

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A linear functional is a linear mapping $\ell : \mathcal{B} \rightarrow \mathbb{R}$, i.e. a mapping such that

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A linear functional ℓ is continuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u, v \in \mathcal{B}$ with $\|u - v\| \leq \delta$ we have $|\ell(u) - \ell(v)| \leq \varepsilon$.

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Note that a space of continuous functionals is a vector space. We can also equip the space of continuous linear functionals over \mathcal{B} with a norm:

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The vector space \mathcal{B}^* is a Banach space.

Linear Functionals and L^p .

Consider $L^p(X, \mathcal{M}, \mu)$ with $p \in [1, \infty]$, as before let q be the dual exponent to p , i.e. $1/p + 1/q = 1$.

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then $\|f\|_\infty = 1$ and $\ell(f) = \int_X f(x)g(x)d\mu = \int_X \text{sign}(g(x))g(x)d\mu = \int_X |g(x)|d\mu = \|g\|_1.$

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$$\|f\|_p^p = \int_X |f(x)|^p d\mu = \int_X \left(\frac{|g(x)|}{\|g\|_q} \right)^{(q-1)p} d\mu = \int_X \left(\frac{|g(x)|}{\|g\|_q} \right)^q d\mu = \frac{1}{\|g\|_q^q} \int_X |g(x)|^q d\mu = 1.$$

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then $\mu(E') \neq 0$, otherwise we could lower the upper bound.

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thus $\|\ell\|^* = \sup_{\|f\|_1 \leq 1} |\ell(f)| \geq \|g\|_\infty - \varepsilon$ and computation was done for all $\varepsilon \geq 0$ we proved

$$\|\ell\|^* \geq \|g\|_\infty.$$

Duality and L^p .

Consider $L^p(X, \mathcal{M}, \mu)$ with $p \in [1, \infty]$, as before let q be the dual exponent to p , i.e. $1/p + 1/q = 1$. We proved

Associating $g \in L^q$ with $\ell = \int_X f(x)g(x)d\mu$ we can create a linear isometry from L^q to a subspace of $(L^p)^*$ (indeed, $\|\ell\|^* = \|g\|_q$). Thus can see L^q and a subspace of $(L^p)^*$!

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Suppose $p \in [1, \infty)$ and $1/p + 1/q = 1$. Then

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in the following sense: for every bounded linear functional ℓ on L^p there is a unique $g \in L^q$ so that

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Suppose $p \in [1, \infty)$ and $1/p + 1/q = 1$. Then

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But first we will prove a lemma showing that it is enough to compute the norm of the linear functional taking the supremum over simple functions only.

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Thus $\|g_n\|_q \leq M$. Now we can apply Fatou lemma to claim that

$$\|g\|_q^q = \int_X |g|^q d\mu \leq \liminf_{n \rightarrow \infty} \int |g_n(x)|^{q-1} |g(x)| d\mu = \liminf_{n \rightarrow \infty} \|g_n\|_q^{q-1} \int_X f_n(x) g(x) d\mu \leq M^q.$$

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Thus ν is countably additive, and applying $|\nu(E)| \leq c\mu(E)^{1/p}$, for all measurable E we get that ν is absolutely continuous with respect to μ !

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The uniqueness of the limit gives us $\ell(f) = \int_X fg d\mu$ and we have finished the proof of the case $\mu(X) < \infty$.

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thus $g \in L^q$. And we can finish the proof by a limiting argument (as before) using the fact that the representation is true for all f supported on E_n and $g \in L^q$.