

MATH- 62052/72052
Functions of Real Variables 2.
Lecture 20.

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Thus for above to be true $\sum \nu(E_j)$ must converge absolutely, i.e. $\sum |\nu(E_j)|$ is convergent.

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$$\nu(E) = \int_E x \, dx,$$

for E Lebesgue measurable in $[-1, 1]$.

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It turns out that for any signed measure ν on (X, \mathcal{M}) we can always find a measure γ on (X, \mathcal{M}) , which would dominate ν , i.e. a measure γ such that

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Taking the supremum over all $\alpha_j < |\nu|(E_j)$ we get $\sum |\nu|(E_j) \leq |\nu|(E)$.

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$$\sum_k |\nu(Q_k)| = \sum_k \left| \sum_j \nu(Q_k \cap E_j) \right|.$$

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An immediate application of above inequality is the fact that if ν is σ -finite, then so are ν^+ and ν^- .