MATH- 62052/72052
Functions of Real Variables 2.
Lecture 21, Part 1.

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Some definitions

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Consider a signed measure ν and a measure (i.e. positive measure) μ on $(X, \mathcal{M})$. We say ν is **absolutely continuous** with respect to μ if

$$\nu(E) = 0 \text{ whenever } E \in \mathcal{M} \text{ and } \mu(E) = 0.$$

We will use symbol $\nu \ll \mu$.

Notice that $m_1 \ll m$ but $m \not\ll m_1$; $\gamma \ll m$ and $m \ll \gamma$. Finally $\delta_0 \not\ll m$ and $m \not\ll \delta_0$.
We also note that if $\nu \ll \mu$ and ν is supported on A (and ν is not a zero measure) then $\mu(A) > 0$ (i.e. A is essential part of support of μ).
Thus if $\nu \perp \mu$ and $\nu \ll \mu$ then ν is identical zero. Indeed if ν is not identical zero then there is $A \in \mathcal{M}$ such that $\nu(A) \neq 0$ but then $\mu(A) > 0$ (this follows from $\nu \ll \mu$).

Some definitions

Consider a measure space $(X, \mathcal{M}, \mu)$. We say that a measure μ is supported on a set $A \in \mathcal{M}$ if $\mu(E) = \mu(E \cap A)$ for all $E \in \mathcal{M}$. For example the measure $m_1 = m(E \cap [-1,1])$ is supported on the interval $[-1,1]$.

Two signed measures ν and μ on $(X, \mathcal{M})$ are **mutually singular** if there are disjoint subsets A and B in $\mathcal{M}$ so that

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A proposition on absolute continuity

In the case of Lebesgue measure m and integrals defined with respect of $f \in L_1(X, m)$, i.e. $\nu(E) = \int_E f dm$ we had a bit stronger assertion of continuity:

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In the next proposition we would like to link the above observation with the case of general measures.

Proposition

Consider a signed measure ν and a measure μ on $(X, \mathcal{M})$.

- Assume that for each $\varepsilon > 0$, there is a $\delta > 0$ such that if $\mu(E) < \delta$ then $|\nu(E)| < \varepsilon$. Then $\nu \ll \mu$.
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where the supremum is taken over all countable partitions of E (i.e. $E = \cup E_j$) into disjoint sets $E_j \in \mathcal{M}$.

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Then $\mu(\bigcup_{k\geq n} E_k) \leq \sum_{k\geq n} 2^{-k} = 2^{-n+1}$. Moreover, all sets $\bigcup_{k\geq n} E_k$ are contained in $\bigcup_{k\geq 1} E_k$, which is of finite measure μ. Thus $\mu(E^*) = 0$. But $\nu(\bigcup_{k\geq n} E_k) \geq \nu(E_n) \geq \varepsilon$. We also assumed that ν is finite.

Consider a signed measure ν and a measure μ on $(X, \mathcal{M})$.

1. If that for each $\varepsilon > 0$, there is a $\delta > 0$ such that if $\mu(E) < \delta$ then $|\nu(E)| < \varepsilon$. Then $\nu \ll \mu$.
2. Conversely, if $\nu \ll \mu$ and $|\nu|$ is a finite measure, then for each $\varepsilon > 0$, there is a $\delta > 0$ such that if $\mu(E) < \delta$ then $|\nu(E)| < \varepsilon$.

Proof: The first statement is trivial. Indeed, take E such that $\mu(E) = 0$, we need to show that $\nu(E) = 0$, but this follows immediately, because if $\mu(E) = 0$, then $\mu(E) < \delta$ for any choice of $\delta > 0$ and thus $|\nu(E)| < \varepsilon$ for all $\varepsilon > 0$.
To prove the converse (i.e. the second statement) it is enough to consider the case when ν is a positive measure. Indeed we can replace ν by $|\nu|$. Now assume, towards a contradiction, that the statement is not true, i.e. that $\nu \ll \mu$, ν finite, but there exists $\varepsilon > 0$ and each $n \in \mathbb{N}$ there exist E_n: $\mu(E_n) < 2^{-n}$ but $\nu(E_n) > \varepsilon$. Let

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$$\nu(E^*) = \lim_{n\to\infty} \nu(\bigcup_{k\geq n} E_k) \geq \varepsilon$$

which is contradiction!