MATH- 62052/72052
Functions of Real Variables 2.
Lecture 21, Part 2.

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Reminders:

- Two signed measures ν and μ on $(X, \mathcal{M})$ are **mutually singular** if there are disjoint subsets A and B in $\mathcal{M}$ so that $\nu(E) = \nu(A \cap E)$ and $\mu(E) = \mu(B \cap E)$ for all $E \in \mathcal{M}$,We will use symbol $\nu \perp \mu$.

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- Consider a signed measure ν and a measure (i.e. positive measure) μ on $(X, \mathcal{M})$. We say ν is **absolutely continuous** with respect to μ if $\nu(E) = 0$ whenever $E \in \mathcal{M}$ and $\mu(E) = 0$. We will use symbol $\nu \ll \mu$.

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There are a number of proofs of the above theorem. We will learn the proof of Neumann, which is based some Hilbert space ideas (and so we justify your time spend playing with Hilbert spaces).

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Let ℓ be a continuous (bounded) linear functional on a Hilbert space $\mathcal{H}$. Then, there exists a unique $g \in \mathcal{H}$ such that

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In our case of $\mathcal{H} = L^2(X, \rho)$ we have $(\phi, g) = \int_X \phi g d\rho$ and, for example $(\chi_E, g) = \int_E g d\rho$ for all $E \in \mathcal{M}$, which "ideologically" looks very much like what we want for ν_a in Radon-Nikodym Theorem. Clearly, all of this just an idea and the proof will be quite more tricky!

Radon-Nikodym Theorem - Proof

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$$\begin{aligned}
|\ell(\phi)| &= \left| \int_X \phi(x) d\nu(x) \right| \leq \int_X |\phi(x)| d\nu(x) \\
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&\leq \left(\int_X |\phi(x)|^2 d\rho(x) \right)^{1/2} \left(\int_X 1^2 d\rho(x) \right)^{1/2} \\
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So using that ρ is a finite measure we get that $|\ell(\phi)| \leq M \|\phi\|_{\mathcal{H}}$ for all $\phi \in \mathcal{H}$.

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Radon-Nikodym Theorem - Proof

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So the proof is finished for the case when ν and μ are finite measures.

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Indeed if we have partition E_j^{μ} for μ and E_i^{ν} for ν we may consider a "joint" partition by $E_{i,j} = E_j^{\mu} \cap E_i^{\nu}$.

Radon-Nikodym Theorem - Proof

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Note that $\mu = \sum \mu_j$ and $\nu = \sum \nu_j$. We can apply our results for finite measures to above measures and get $\nu_j = \nu_{j,a} + \nu_{j,s}$ where $\nu_{j,s} \perp \mu_j$ and $\nu_{j,a} = f_j d\mu_j$. Then it is sufficient to set

$$f = \sum f_j, \ \nu_s = \sum \nu_{j,s} \text{ and } \nu_a = \sum \nu_{j,a},$$

to finish the proof for the case of two σ-finite measures.
If ν is a signed measure, decompose it to positive and negative variation (i.e. $\nu = \nu^+ - \nu^-$) and apply the above argument to ν^+ and ν^-.
Finally, we need to prove the uniqueness part. Assume

$$\nu = \nu_a + \nu_s = \nu_a' + \nu_s'$$

where $\nu_a \ll \mu$ and $\nu_a' \ll \mu$; $\nu_s \perp \mu$ and $\nu_s' \perp \mu$. Then

$$\nu_a - \nu_a' = \nu_s' - \nu_s,$$

but $(\nu_a - \nu_a') \ll \mu$ and $(\nu_s' - \nu_s) \perp \mu$,

If μ and ν are σ-finite measures we may assume that there is the same partition $E_j \in \mathcal{M}$, $X = \cup E_j$

$$\mu(E_j) < \infty \text{ and } \nu(E_j) < \infty.$$

Moreover, making further sub-partition we may assume that E_j are disjoint. So now we may define a finite measures on $\mathcal{M}$ by

$$\mu_j(E) = \mu(E \cap E_j) \text{ and } \nu_j(E) = \nu(E \cap E_j).$$

Note that $\mu = \sum \mu_j$ and $\nu = \sum \nu_j$. We can apply our results for finite measures to above measures and get $\nu_j = \nu_{j,a} + \nu_{j,s}$ where $\nu_{j,s} \perp \mu_j$ and $\nu_{j,a} = f_j d\mu_j$. Then it is sufficient to set

$$f = \sum f_j, \ \nu_s = \sum \nu_{j,s} \text{ and } \nu_a = \sum \nu_{j,a},$$

to finish the proof for the case of two σ-finite measures.
If ν is a signed measure, decompose it to positive and negative variation (i.e. $\nu = \nu^+ - \nu^-$) and apply the above argument to ν^+ and ν^-.
Finally, we need to prove the uniqueness part. Assume

$$\nu = \nu_a + \nu_s = \nu'_a + \nu'_s$$

where $\nu_a \ll \mu$ and $\nu'_a \ll \mu$; $\nu_s \perp \mu$ and $\nu'_s \perp \mu$. Then

$$\nu_a - \nu'_a = \nu'_s - \nu_s,$$

but $(\nu_a - \nu'_a) \ll \mu$ and $(\nu'_s - \nu_s) \perp \mu$, THUS BOTH $(\nu_a - \nu'_a)$ and $(\nu'_s - \nu_s)$ are zero!!!

□