MATH- 62052/72052
Functions of Real Variables 2.
Lecture 22.

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Department of Mathematical Sciences, Kent State University

Spring, 2020

The Abel Prize 2020

"The Abel Prize was established on 1 January 2002. The purpose is to award the Abel Prize for outstanding scientific work in the field of mathematics. The prize amount is 7,5 million Norwegian Krone (about 715,000 US dollars) and was awarded for the first time on 3 June 2003.
The prize is awarded by the Norwegian Academy of Science and Letters, which has appointed an Abel Committee consisting of five mathematicians to review the nominated candidates and submit a recommendation for a worthy Abel laureate."

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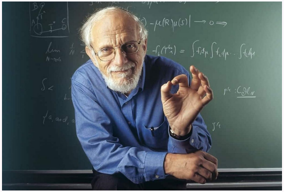
This Year



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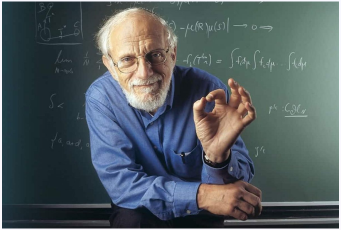

Hillel (Harry) Furstenberg (born in Germany, in 1935. In 1939, Shortly after Kristallnacht, his family escaped to the United States) is an American-Israeli mathematician and professor emeritus at the Hebrew University of Jerusalem. He is a member of the Israel Academy of Sciences and Humanities and U.S. National Academy of Sciences and a laureate of the Abel Prize and the Wolf Prize in Mathematics. He is known for his application of probability theory and **ergodic theory** methods to other areas of mathematics, including number theory and Lie groups.

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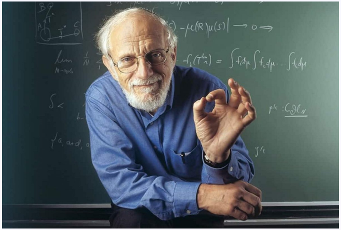

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In June 2017, Analysis group at Kent State have organized NSF supported conference "Ergodic Methods in the Theory of Fractals", during which Professor Furstenberg delivered a series of ten lectures. As an outcome of this lecture series a book titled by "Ergodic Theory and Fractal Geometry" by Hillel Furstenberg was co-published by AMS and CBMS.

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Very basic fact

τ - measure preserving, f- measurable, then $f(\tau(x))$ is measurable and

$$\int_X f(\tau(x)) d\mu(x) = \int_X f(x) d\mu(x).$$

This follows from $\{x \in X : f(\tau(x)) > t\} = \tau^{-1}(\{y \in X : f(y) > t\})$ and thus is measurable.

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$$\mu(\{x \in X : f(\tau(x)) > t\}) = \mu(\{y \in X : f(y) > t\}).$$

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Plug $f = \chi_E$, $E = \{1\}$ and $\tau(n) = 2n$:

$$\int_{\mathbb{Z}} \chi_E(\tau(n))d\mu(n) = \mu\{n \in \mathbb{Z} : \tau(z) \in \{1\}\} = 0 \neq 1 = \mu(E) = \int_{\mathbb{Z}} \chi_E(n)d\mu(n).$$

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We can generalize the above example to the torus $\mathbb{T}^n = (0,1]^n$ with Lebesgue measure on its Borel σ-algebra, and, as before consider addition modulo 1 in each coordinate:

$$\tau(x_1, ..., x_n) = (x_1 + a_1, ..., x_n + a_n),$$

where $a = (a_1, \dots, a_n) \in \mathbb{T}^n$.

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Let $\mathcal{B}$ be a σ-algebra generated by an algebra $\mathcal{A}$, then μ is preserved by a measurable transformation τ if and only if

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Proof: Assume we have that $\mu(\tau^{-1}(A)) = \mu(A)$, for all $A \in \mathcal{A}$, then $\mu = \mu(\tau^{-1})$ on $\mathcal{A}$, but then we can use the uniqueness part of extension theorem from algebra $\mathcal{A}$ to σ-algebra $\mathcal{B}$ to declare that $\mu = \mu(\tau^{-1})$ on $\mathcal{B}$ and thus τ is measure preserving!

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i.e., it is enough to check the measure preserving relation for the elements on the generating algebra $\mathcal{A}$ and then it automatically holds for all elements of $\mathcal{B}$.

Proof: Assume we have that $\mu(\tau^{-1}(A)) = \mu(A)$, for all $A \in \mathcal{A}$, then $\mu = \mu(\tau^{-1})$ on $\mathcal{A}$, but then we can use the uniqueness part of extension theorem from algebra $\mathcal{A}$ to σ-algebra $\mathcal{B}$ to declare that $\mu = \mu(\tau^{-1})$ on $\mathcal{B}$ and thus τ is measure preserving!
The converse is trivial, clearly if μ is measure preserving on $\mathcal{B}$ is is measure preserving on a subset of $\mathcal{B}$ (and thus on $\mathcal{A}$).

$\square$

A useful fact

Consider a transformation τ acting on $(X, \mathcal{M}, \mu)$. We say that τ is measurable if $\tau^{-1}(E)$ is measurable for every $E \in \mathcal{M}$. Note that in this case we can define a new measure $\mu(\tau^{-1}) : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ acting on $E \in \mathcal{M}$ simply by $\mu(\tau^{-1}(E))$ (check that $\mu(\tau^{-1})$ is a measure!) and is called push-forward of μ with respect to τ. Thus τ is measure preserving transformation iff $\mu(\tau^{-1}) = \mu$ on $\mathcal{M}$.

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This facts makes checking three previous examples in particular easy, indeed, it is enough to check that a measure of a segment (or box) is preserved.

Measure preserving transformations: Examples

A bit less trivial example on $\mathbb{T}$

$\mathbb{T} = \mathbb{R}/\mathbb{Z}$, i.e. we identify all real numbers mod 1 and create a unit circle. μ is a a measure induced from the Lebesgue measure on $\mathbb{R}$, i.e. $\mathbb{T} = (0,1]$ and μ is m restricted to $(0,1]$. Let $\tau(x) = 2x$ mod 1.

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From here it is easy to continue the argument to the union of segments and use the extension theorem (in short a fact on the previous slide) from Algebra to σ-algebra. *A small side note:* Please, note again that $\mu(\tau(0,1/2])=\mu((0,1])=1\neq\mu((0,1/2])$ which again explains the use of τ^{-1}.

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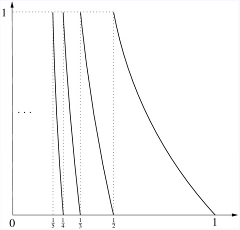

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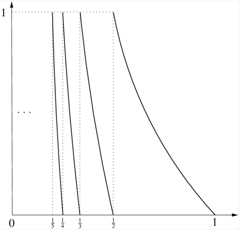

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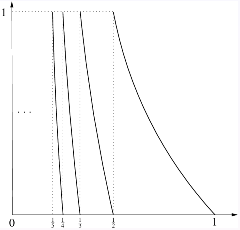

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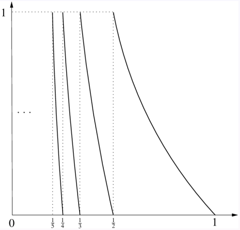

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In particular, $\mu([0,1]) = 1$ and thus μ is a probability measure on $[0,1]$.

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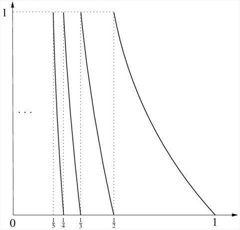

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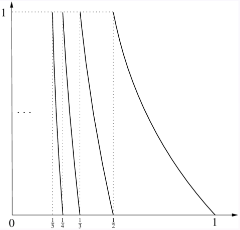

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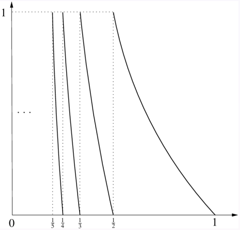

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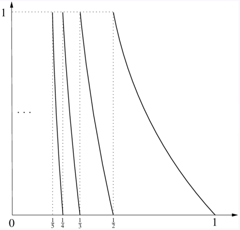

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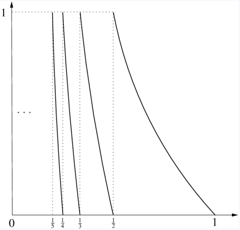

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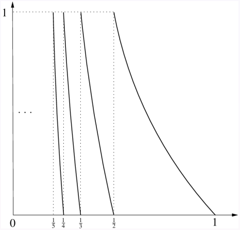

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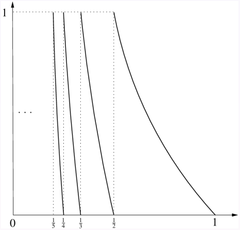

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Now we are left with direct computation. Using that $\{G_n^{-1}([a,b])\}_{n=1}^{\infty}$ are disjoint we get

$$\mu(G^{-1}([a,b])) = \mu(\cup G_n^{-1}([a,b])) = \sum \mu(G_n^{-1}([a,b])) = \sum \mu\left(\left[\frac{1}{b+n}, \frac{1}{a+n} \right] \right).$$

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$$=\frac{1}{\log 2}\sum\left[\log\left(\frac{1+a+n}{a+n}\right)-\log\left(\frac{1+b+n}{b+n}\right)\right]$$

Using tha the above sums are telescoping we get

$$\sum_{n=1}^{N}\log\left(\frac{1+a+n}{a+n}\right)=\sum_{n=1}^{N}[\log(1+a+n)-\log(a+n)]=\log(1+a+N)-\log(a+1)$$

so

$$\mu(G^{-1}([a,b]))=\frac{1}{\log 2}\lim_{N\to\infty}[\log(1+a+N)-\log(a+1)-\log(1+b+N)+\log(b+1)]$$

$$=\frac{1}{\log 2}\lim_{N\to\infty}\left[\log\frac{1+a+N}{1+b+N}+\log\frac{b+1}{a+1}\right]=\frac{1}{\log 2}\left[\log\frac{b+1}{a+1}\right]=\mu([a,b]).$$

So we proved that G is measure preserving on intervals, thus applying extension fact from above we get that G is a measure preserving transformation.

□