MATH- 62052/72052
Functions of Real Variables 2.
Lecture 23.

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Measure preserving transformations

We will play in σ-finite measure space $(X, \mathcal{M}, \mu)$. One of our heroes will be

Measure-preserving transformation:

A mapping
$$\tau : X \to X \text{ such that } \mu(\tau^{-1}E) = \mu(E) \text{ for all } E \in \mathcal{M},$$
here $\tau^{-1}(E)$ is a pre-image of E, i.e. $\tau^{-1}(E) = \{x \in X : \tau(x) \in E\}$.

Also if τ is measure-preserving transformation + bijection + τ^{-1} is measure-preserving transformation, then τ is **measure-preserving isomorphism**.

Very basic fact

τ - measure preserving, f- measurable, then $f(\tau(x))$ is measurable and
$$\int_X f(\tau(x))d\mu(x) = \int_X f(x)d\mu(x).$$

$$\mu(\{x \in X : f(\tau(x)) > t\}) = \mu(\{y \in X : f(y) > t\}).$$

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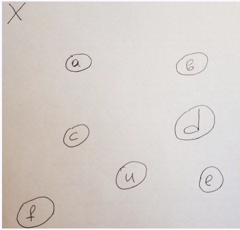

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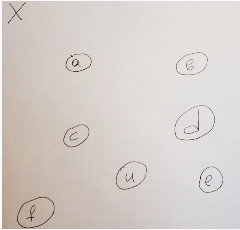

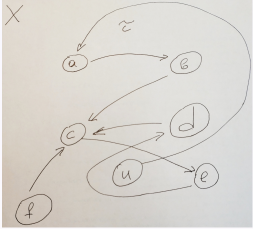

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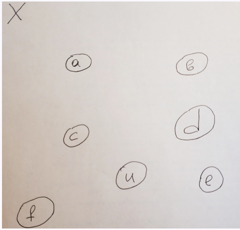

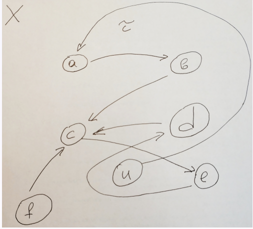

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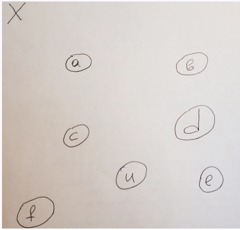

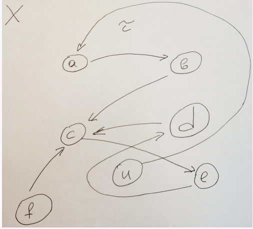

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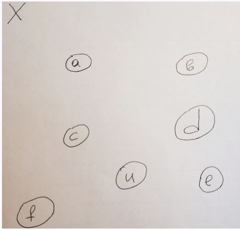

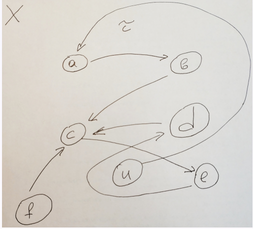

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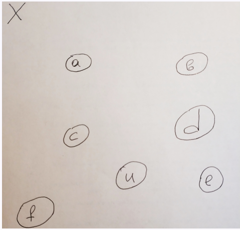

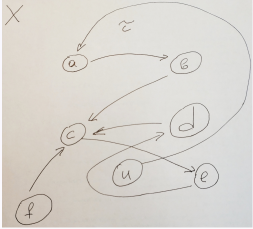

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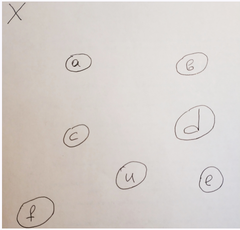

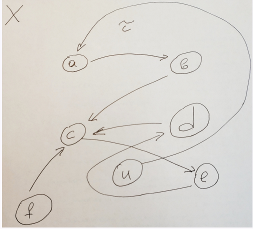

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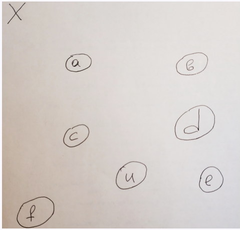

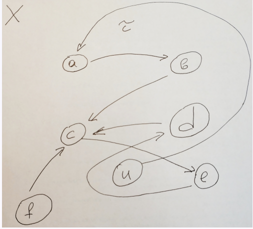

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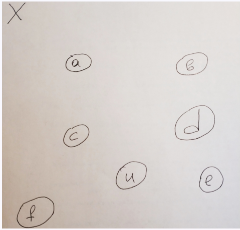

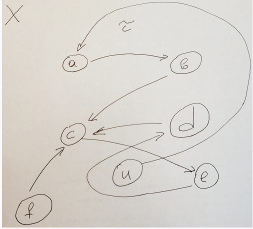

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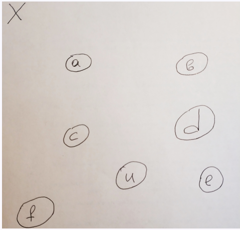

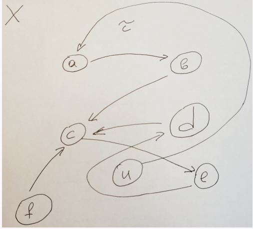

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$$A_n(f)(x) = \frac{1}{n} \sum_{k=0}^{n-1} f(\tau^k(x)),$$

as well as the limits of $A_n(f)(x)$ as $n \to \infty$

Hilbert space approach

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Mean Ergodic Theorem

Suppose T is an isometry of the Hilbert space $\mathcal{H}$, and let P be the orthogonal projection on the subspace of the invariant vectors of T. Let

$$A_n = \frac{1}{n}(I + T + T^2 + \cdots + T^{n-1}).$$

Then for each $f \in \mathcal{H}$, $A_n(f)$ converges to $P(f)$ in norm of $\mathcal{H}$ as $n \to \infty$.

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We will first need to prove a nice lemma about a structure of subspace S.

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$$(Ta, b) = \sum_{i=2}^{\infty} a_{i-1} b_i = \sum_{i=1}^{\infty} a_i b_{i+1} = (a, T^*b),$$

where

$$T^*(b) = (b_2, b_3, b_4, \dots.)$$

is not an isometry (indeed, $\|T(b)\|^2 = \|b\|^2 - b_1^2$).

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Now with the above Lemma in our tool box we are ready to prove Mean ergodic theorem.

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Suppose T is an isometry of the Hilbert space $\mathcal{H}$, and let P be the orthogonal projection on the subspace of the invariant vectors of T. Let

$$A_n = \frac{1}{n}(I + T + T^2 + \cdots + T^{n-1}).$$

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Using that T is an isometry we get

$$\|A_n(f_1')\| = \frac{1}{n}\|(g - T^n g)\| \le \frac{1}{n}(\|g\| + \|T^n g)\|) = \frac{2\|g\|}{n}$$

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the above it true for all $\varepsilon > 0$

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$$A_n = \frac{1}{n}(I + T + T^2 + \dots + T^{n-1}).$$

Then for each $f \in \mathcal{H}$, $A_n(f)$ converges to $P(f)$ in norm of $\mathcal{H}$ as $n \to \infty$.

Proof: Consider $f \in \mathcal{H}$. Using that $\mathcal{H} = \overline{S_1} \oplus (\overline{S_1})^{\perp} = \overline{S_1} \oplus S$, write $f = f_0 + f_1$ where $f_0 \in S$ and $f_1 \in \overline{S_1}$. To work with vectors in S_1 (instead of $\overline{S_1}$) fix $\varepsilon > 0$ and consider $f_1' \in S_1$: $\|f_1 - f_1'\| \leq \varepsilon$:

$$A_n(f) = A_n(f_0) + A_n(f_1') + A_n(f_1 - f_1').$$

Finally, for $A_n(f_1 - f_1')$ we again use that T^k is an isometry:

$$\|A_n(f_1 - f_1')\| = \frac{1}{n}\|\sum_{k=0}^{n-1} T^k(f_1 - f_1')\| \leq \frac{1}{n}\sum_{k=0}^{n-1}\|T^k(f_1 - f_1')\| = \frac{1}{n}\sum_{k=0}^{n-1}\|f_1 - f_1'\| \leq \varepsilon.$$

Putting those three computations together we get

$$\limsup_{n\to\infty}\|A_n(f) - P(f)\| \leq \varepsilon$$

the above it true for all $\varepsilon > 0$ and thus

$$\lim_{n\to\infty}\|A_n(f) - P(f)\| = 0.$$

□