MATH- 62052/72052
Functions of Real Variables 2.
Lecture 25.

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We will play in σ-finite measure space $(X, \mathcal{M}, \mu)$. One of our heroes will be

Measure-preserving transformation:

A mapping
$$\tau : X \to X \text{ such that } \mu(\tau^{-1}E) = \mu(E) \text{ for all } E \in \mathcal{M},$$
here $\tau^{-1}(E)$ is a pre-image of E, i.e. $\tau^{-1}(E) = \{x \in X : \tau(x) \in E\}$.

Also if τ is measure-preserving transformation + bijection + τ^{-1} is measure-preserving transformation, then τ is **measure-preserving isomorphism**.

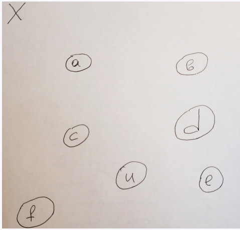
Very basic fact

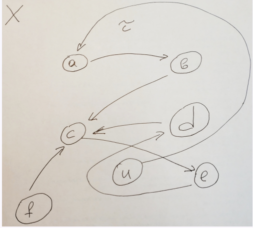
τ - measure preserving, f- measurable, then $f(\tau(x))$ is measurable and
$$\int_X f(\tau(x))d\mu(x) = \int_X f(x)d\mu(x).$$

$$\mu(\{x \in X : f(\tau(x)) > t\}) = \mu(\{y \in X : f(y) > t\}).$$

Transformations and time

We can imagine our space X to be a set of "states" of some system S.





At each (discrete) time n our system at some state $x \in X$ (i.e. $S(n) = x$). What we know that at time $n+1$ our system is at state $S(n+1) = \tau(S(n))$, where the mapping $\tau : X \to X$ presents a transformation of our system after a unit of time, i.e. the next step of the system $(S(n+1))$ depends only on "position" of the system $(S(n))$ and independent of the moment of time when the step is taken.

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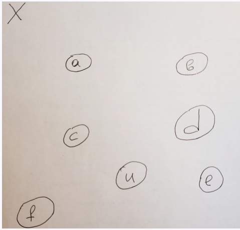

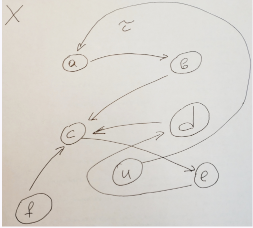

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So if we started at point $x \in S$ then after k steps we will be at point

$$\tau \circ \tau \circ \cdots \circ \tau(x) = \tau^k(x).$$

Thus the mapping τ^k describe the evolution of the system after k units of time. It is interesting to understand the limiting behavior of the system as well as the average behavior of the system and different parameters associated to it.

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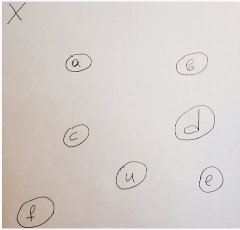

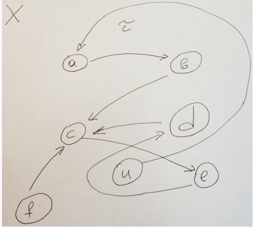

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Thus the mapping τ^k describe the evolution of the system after k units of time. It is interesting to understand the limiting behavior of the system as well as the average behavior of the system and different parameters associated to it. Last time we discussed

$$A_n(f)(x) = \frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k(x)),$$

as well as the limits of $A_n(f)(x)$ as $n \to \infty$ and applied what we learned about Hilbert spaces.

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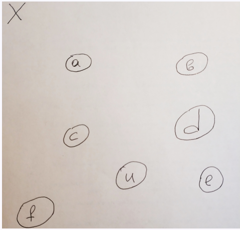

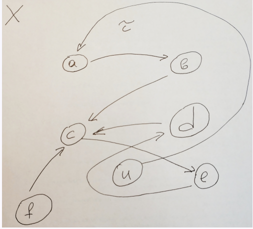

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Thus the mapping τ^k describe the evolution of the system after k units of time. It is interesting to understand the limiting behavior of the system as well as the average behavior of the system and different parameters associated to it. Today we will apply "maximal function techniques" and study

$$f^*(x) = \sup_{1 \le m < \infty} A_m(f) = \sup_{1 \le m < \infty} \frac{1}{m} \sum_{k=0}^{m-1} f(\tau^k(x)),$$

Two theorems we proved

Consider a Hilber space $\mathcal{H} = L^2(X, \mathcal{M}, \mu)$. Let $\tau : X \to X$ be a measure preserving transformation. Define a linear operator $T(f)(x) = f(\tau(x))$.
Reminder T is isometry, indeed $\|Tf\|^2 = \int_x f^2(\tau(x))d\mu(x) = \int_x f^2(x)d\mu(x) = \|f\|^2$. Rewrite:

$$A_n(f)(x) = \frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k(x)) = \frac{1}{n}\sum_{k=0}^{n-1} T^k f(x).$$

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Mean Ergodic Theorem

Suppose T is an isometry of the Hilbert space $\mathcal{H}$, and let P be the orthogonal projection on the subspace of the invariant vectors of T. Let

$$A_n = \frac{1}{n}(I + T + T^2 + \cdots + T^{n-1}).$$

Then for each $f \in \mathcal{H}$, $A_n(f)$ converges to $P(f)$ in norm of $\mathcal{H}$ as $n \to \infty$.

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Maximal function approach: $f^*(x) = \sup\limits_{1 \le m < \infty} A_m(f) = \sup\limits_{1 \le m < \infty} \frac{1}{m}\sum_{k=0}^{m-1} f(\tau^k(x))$.

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Assume $f \in L^1(X, \mu)$ and τ is a measure preserving transformation, then the maximal function $f^*(x)$ is finite for almost every x. Moreover,

$$\mu(\{x : f^*(x) > \alpha\}) \le \frac{6}{\alpha}\|f\|_{L^1(X,\mu)}, \text{ for all } \alpha > 0.$$

Now we are ready for pointwise convergence.

We will consider probability space $(X, \mathcal{M}, \mu)$ (i.e. $\mu(X) = 1$).

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$$\int_X |\tilde{P}(f)(x)| d\mu(x) \leq \int_X |f(x) d\mu(x).$$

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Indeed, if $f \in L^1(X)$ then

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But (from the Theorem) $A_m(f)(x)$ converges to $\tilde{P}(f)$ almost everywhere and we can apply the Fatou's lemma.

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Then $\int_X |f(x)|^2 d\mu(x) \leq \int_X n^2 d\mu(x) \leq n^2$ and this $f_n \in L^2(X, \mu)$ and $\|f - f_n\|_1 \to 0$ by dominated convergence theorem (indeed $f(x) - f_n(x) \to 0$ almost every where and $|f(x) - f_n(x)| \leq f(x)$.

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$$f = F + H, \text{ where } \|H\|_{L^1} < \varepsilon, \text{ and } F = F_0 + (I - T)G,$$

where $F_0, G \in L^2$, $T(F_0) = F_0$ (remember T is our old friend - $T(f)(x) = f(\tau(x))$).

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$$f = f' + h' = F_o + F_1 + (h + h') = F_o + (I - T)G + H,$$

where $H = h + h'$ and $\|H\|_{L^1} \le \|h\|_{L^1} + \|h'\|_{L^1} \le \varepsilon$.

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We know that $A_n(F)$ is already convergent almost everywhere, thus satisfy Cauchy criterium almost everywhere, so E_α is contained (up to measure zero) in

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and to finish the proof we need to estimate that $\mu(E'_\alpha)$.

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Proof: We have seen that every function $f \in L^1(X, \mu)$ can be approximated by functions from $L^2(X, \mu)$. We need to be a bit more precise and constructive. We claim that for every $\varepsilon > 0$ we can write

$$f = F + H, \text{ where } \|H\|_{L^1} < \varepsilon, \text{ and } F = F_0 + (I - T)G,$$

where $F_0, G \in L^2$, $T(F_0) = F_0$. (remember T is our old friend - $T(f)(x) = f(\tau(x))$. $A_m(F)$ converges to F_0 for a.e. x. To prove convergence for $A_m f(x)$ define

$$E_\alpha = \{x : \lim_{N \to \infty} \sup_{n,m \geq N} |A_n(f)(x) - A_m(f)(x)| > \alpha\}.$$

Our goal is to show that $\mu(E_\alpha) = 0$ for all $\alpha > 0$. Let

$$E'_\alpha = \{x : \lim_{N \to \infty} \sup_{n,m \geq N} |A_n(H)(x) - A_m(H)(x)| > \alpha\}$$

then $\mu(E_\alpha) \leq \mu(E'_\alpha)$.

Now we are ready for pointwise convergence.

We will consider probability space $(X, \mathcal{M}, \mu)$ (i.e. $\mu(X) = 1$).

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$$\mu(\{x : \lim_{N\to\infty} \sup_{n,m \geq N} |A_n(H)(x) - A_m(H)(x)| > \alpha\}) \leq \mu(\{x : 2\sup |A_m(H)(x)| > \alpha\})$$

$$= \mu(\{x : \sup |A_m(H)(x)| > \alpha/2\}) \leq \frac{6}{\alpha/2} \|H\|_{L_1} \leq \frac{12\varepsilon}{\alpha},$$

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$$= \mu(\{x : \sup |A_m(H)(x)| > \alpha/2\}) \leq \frac{6}{\alpha/2}\|H\|_{L_1} \leq \frac{12\varepsilon}{\alpha},$$

where ε is arbitrary so $\mu(E_\alpha) = 0$.