MATH- 62052/72052
Functions of Real Variables 2.
Lecture 26.

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Spring, 2020

We will play in σ-finite measure space $(X, \mathcal{M}, \mu)$. One of our heroes will be

Measure-preserving transformation:

A mapping

$$\tau : X \to X \text{ such that } \mu(\tau^{-1}E) = \mu(E) \text{ for all } E \in \mathcal{M},$$

here $\tau^{-1}(E)$ is a pre-image of E, i.e. $\tau^{-1}(E) = \{x \in X : \tau(x) \in E\}$.

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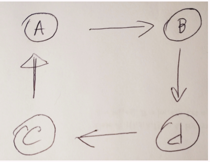

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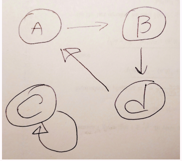
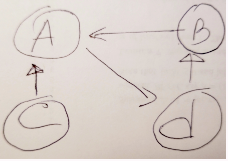


Not Ergodic



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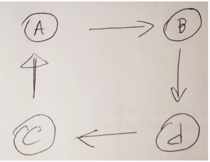

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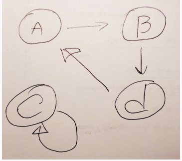

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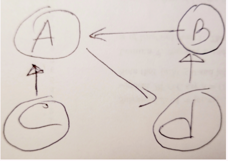

Not Ergodic

Can ergodic transformation have fixed points? Sets? YES! (for example rotation in $\mathbb{R}^d$ or reflection about a plane in $\mathbb{R}^d$) but the the measure of a "fixed set" must be zero OR it should be "about all " X!

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Pointwise convergence + Ergodicity.

Consider probability space $(X, \mathcal{M}, \mu)$ (i.e. $\mu(X) = 1$).

Pointwise Ergodic Theorem

Consider $f \in L^1(X, \mu)$. Then for almost evert $x \in X$ we have that $A_m(f)(x) = \frac{1}{m}\sum_{k=0}^{m-1} f(\tau^k(x))$ converge to a limit as $m \to \infty$

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Remark: If $f \in L^2(X, \mu)$ then the limit must be $P(f)$, where P is the orthogonal projection on the subspace of the invariant vectors of operator $Tf = f(\tau(x))$.

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$$P(f) = (1, f) \times 1 = \int_X f d\mu(x).$$

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Remark: If $f \in L^2(X, \mu)$ then the limit must be $P(f)$, where P is the orthogonal projection on the subspace of the invariant vectors of operator $Tf = f(\tau(x))$. Now assume τ is Ergodic. Then the only invariant vectors of T are constants!! So we can simply compute the orthogonal projection (the basis vector of the subspace of invariant vectors is simply $f(x) = 1$):

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Indeed, if we denote by S the subspace of invariant vectors of T, then S consists of constants only. Thus a function 1 is the basis unit vector (we are in probability space $\int_X 1^2 d\mu = 1$). Using that $L_2(X, \mu) = S \oplus S^{\perp}$ we can write $f = P(f) + g$, where $g \in S^{\perp}$:

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Examples: Rotations of the circle.

$\mathbb{T} = \mathbb{R}/\mathbb{Z}$, i.e. we identify all real numbers mod 1 and create a unit circle. μ is a a measure induced from the Lebesgue measure on $\mathbb{R}$, i.e. $\mathbb{T} = (0, 1]$ and μ is m restricted to $(0, 1]$. Fix $\alpha \in \mathbb{R}$ and let $\tau(x) = x + \alpha$ mod 1 (i.e. it just a rotation of our circle by $2\pi\alpha$).

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Assume P is a projection on τ invariant L^2 functions. τ is a measure preserving transformation, so by mean ergodic theorem

$$A_m(f)(x) = \frac{1}{m} \sum_{k=0}^{m-1} f(\tau^k(x)) = \frac{1}{m} \sum_{k=0}^{m-1} f(x+k\alpha)$$

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Now assume τ is ergodic. Suppose $\alpha = p/q$ is rational. Take $E_0 \in (0, 1/q)$ such that $0 < m(E_0) < 1/q$ (for example $E_0 = (0, 1/(2q))$). Let E be a disjoint union of $\cup_{r=0}^{q-1}(E_0 + r/q)$. Then $\tau(E) = E$ but $m(E) = rm(E_0)$ and thus $m(E) > 0$ and $m(E^c) = 1 - m(E) > 0$ and τ is not ergodic.

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We note that NOT every ergodic map is uniquely ergodic. We will contract an example on the next slides.

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We will show that τ not only ergodic, but satisfies a stronger property:

τ is **mixing**, if τ is measure preserving on (X, μ) and

$$\mu(\tau^{-n}(E) \cap F) \to \mu(E)\mu(F), \text{ as } n \to \infty, \text{ for all measurable } E, F.$$

Examples: The doubling mapping on the circle.

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We also note that the reverse implication (that mixing implied the condition on operator T) can be proved via approximation by simple functions (have fun with Problem 3 in the Homework).

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Thus $(P(f), g) = (f, 1)(1, g)$.

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this integral is 1 if $m = k = 0$, otherwise, it is 0 if n is large enough. Thus we have shown the condition for exponential functions and by linearity for all trigonometric polynomials. Thus the condition follows by approximation for all $f, g \in L^2$.

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