RECONSTRUCTION OF CONVEX BODIES OF REVOLUTION FROM THE AREAS OF THEIR SHADOWS

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Abstract. In this note we reconstruct a convex body of revolution from the areas of its shadows by giving a precise formula for the support function.

1. Introduction

The problem of reconstruction of a convex body from the areas of its shadows, goes back to A. D. Aleksandrov [Al1], who proved that an origin symmetric convex body K in $\mathbb{R}^n$ is uniquely defined by the volumes of its projections. Recently R. Gardner and P. Milanfar [GM] provided an algorithm for reconstruction of an origin-symmetric convex body K from the volumes of its projections.

It is plausible that there exist an explicit formula, connecting the support function of K with volumes of its shadows. The similarities between sections and projections, pointed out in [KRZ2], suggest that it should exist as a dual version of the formula for sections, proved by A. Koldobsky [K]

$$\mathrm{Vol}_{n-1}(K\cap\theta^\perp)=\frac{1}{\pi(n-1)}(\|\cdot\|_K^{-n+1})^\wedge(\theta),\ \theta\in S^{n-1}. \tag{1}$$

Note that by inverting the Fourier transform in (1), one can find the direct formula for the norm of K, given the volumes of sections.

It was proved in [KRZ1] that there is a connection between the volumes of projections of K and the curvature function f_K via the Fourier transform:

$$\mathrm{Vol}_{n-1}(K|\theta^\perp)\ =\ -\frac{1}{\pi}\widehat{f_K}(\theta),\qquad \forall\theta\in S^{n-1}. \tag{2}$$

Here $f_K(x)\ =\ |x|^{-n-1}f_K(x/|x|),\ x\in\mathbb{R}^n\setminus\{0\}$, is the extension of $f_K(x)$, $x\in S^{n-1}$, to a homogeneous function of degree $-n-1$.

2000 *Mathematics Subject Classification.* Primary: 52A15, 52A21, 52A38.

Key words and phrases. Convex body, geometric tomography, surface area measure, cosine transform, Alexandrov's theorem.

A. D. Aleksandrov ([Al2]; [Pog], p. 456; [Sc], Corollary 2.5.3) showed that f_K is the sum $\Sigma(h_K)$ of the principal minors of order $n-1$ of the Hessian matrix of the support function h_K. This suggests a dual version of (1)

$$\mathrm{Vol}_{n-1}(K|\theta^{\perp}) = -\frac{1}{\pi}\widehat{\Sigma(h_K)}(\theta), \qquad \forall\theta \in S^{n-1}, \tag{3}$$

which in the three dimensional case has the form:

$$\mathrm{Vol}_2(K|\theta^{\perp}) = -\frac{1}{\pi}\left(\left|\begin{array}{cc} h_{xx} & h_{xy} \\ h_{xy} & h_{yy}\end{array}\right| + \left|\begin{array}{cc} h_{xx} & h_{xz} \\ h_{xz} & h_{zz}\end{array}\right| + \left|\begin{array}{cc} h_{yy} & h_{zy} \\ h_{zy} & h_{zz}\end{array}\right|\right)^{\wedge}(\theta).$$

Unfortunately, to invert the above formulas one needs not only to invert the Fourier transform, but also to solve a nonlinear differential equation. It turns out that in the case of a body of revolution, this differential equation can be considerably simplified. In Section 2 we show how to obtain an expression for h_K, given the curvature function f_K. In Section 3 we give a simple formula for the curvature function via the projections of K (cf. [R], p. 125). All arguments can be generalized to higher dimensions.

Let $r \geq$ be a natural number. A real valued function on an open subset U of $\mathbb{R}^3$ is said to be *of class* C^r (cf. [Ga3], p. 22) if it is $r-$times differentiable, that is all partial derivatives of order r exist and are continuous. We denote this class by $C^r(U)$. A function $f(\sigma)$ on $\sigma \in S^2$ is said to be in $C^r(S^2)$ if its homogeneous extension

$$f(\frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}) \in C^r(\mathbb{R}^3 \setminus \{0\}).$$

We say that a convex body K *is of class* C^r (cf. [Ga3], p. 23) if ∂K is of class C^r as a submanifold of $\mathbb{R}^3$. If $k \geq 2$, we say that K is of class C^k_+ (cf. [Ga3], p. 25), if K is of class C^k and the Gauss curvature of K at each point is positive.

Without loss of generality we may assume that e_3 is the axis of revolution. Our main result is the following

Theorem 1. *Let* $(x,y,z) \in S^2$, *and let* K *be of class* C^5_+. *Then*

$$h_K(x,y,z) = \sqrt{x^2+y^2}\ \ \phi(\arcsin|z|) + |z| \int\limits_0^{\arcsin|z|} \frac{\cos^2 t f(t)}{\phi(t)}\,dt,$$

where

$$\phi(t) = \sqrt{\int\limits_t^{\pi/2} \sin(2\alpha)\, f(\alpha) d\alpha},$$

$$f(\alpha) = f_K(0, \cos\alpha, \sin\alpha) = \frac{\sin\alpha}{6} \int\limits_0^{\sin\alpha} \frac{ds}{(\sin^2\alpha - s^2)^{\frac{1}{2}}} \times$$

$$\frac{d}{ds}\left[\frac{1}{s}\frac{d}{ds}\left(s(1-s^2)^{3/2}\frac{d}{ds}\left(\frac{\mathrm{Vol}_2(K|(0,s,\sqrt{1-s^2})^{\perp})}{\sqrt{1-s^2}}\right)\right)\right],$$

where $\alpha \in [0, \pi/2]$.

We would like to make a remark concerning the smoothness hypothesis in our theorem. It is clear that if the function

$$\mathrm{Vol}_2(K|\cdot^{\perp}) : u \in S^2 \to \mathrm{Vol}_2(K|u^{\perp})$$

belongs to $C^3(S^2)$, and is rotation-invariant, then the function

$$\mathrm{Vol}_2(K|(0,\cdot,\sqrt{1-\cdot^2})^{\perp}) : s \in (0,1) \to \mathrm{Vol}_2(K|(0,s,\sqrt{1-s^2})^{\perp})$$

belongs to $C^3((0,1))$. Thus, it is enough to assume that K is such that $\mathrm{Vol}_2(K|\cdot^{\perp}) \in C^3(S^2)$. This is true, provided $f_K \in C^3(S^2)$ (see Lemma 4). One can weaken this hypothesis, but this is not our purpose here (see [M]).

We also remark that in many problems of convexity the bodies of revolution serve as a main source for examples and counterexamples, see for example the Shephard problem [P], ([Ga3], p. 142), or the Busemann-Petty problem [Ga1], [Ga2], [Pa]. Therefore, different types of special formulas for bodies of revolution may lead to the general development of the techniques related to the more general classes of convex bodies (cf. [Ga2] and [GKS]). We hope that the formulas obtained in Lemma 1 and Theorem 1 will help to provide new connections between the volumes of sections (1) and projections (2) of general convex bodies.

2. From the curvature to the support function

We recall that the curvature function f_K is the reciprocal of the Gauss curvature viewed as a function of the unit normal vector ([Sc], p. 419). The support function of the convex body K is defined as $h_K(\xi) = \sup\{\eta \cdot \xi,\ \eta \in K\}$. It is proved (see [Sc], pp. 106-111) that K is of class C^2_+ if and only if $h_K \in C^2$ and the Gauss curvature of K exists and is positive everywhere.

It is enough to consider the case $x, y, z \geq 0$ (all other cases can be reconstructed by symmetry). We will use the following notation $f_K(x,y,z) = f_K(u,v)$, $u = \sqrt{x^2+y^2}$ and $v = z$, and our goal is to find $h_K(x,y,z) = h(u,v)$. We will need three elementary lemmas.

Lemma 1. *Let K be a body of revolution, then the equation*

$$f_K = \begin{vmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{vmatrix} + \begin{vmatrix} h_{xx} & h_{xz} \\ h_{xz} & h_{zz} \end{vmatrix} + \begin{vmatrix} h_{yy} & h_{zy} \\ h_{zy} & h_{zz} \end{vmatrix}$$

has the form

$$f_K(u,v) = \frac{h_u}{u}\left(h_{uu} + h_{vv}\right), \qquad u^2 + v^2 = 1.$$

Proof : A straightforward computation gives

$$h_x = h_u \frac{x}{\sqrt{x^2+y^2}},$$

$$h_{xx} = h_{uu}\frac{x^2}{x^2+y^2} + h_u \frac{y^2}{(x^2+y^2)^{3/2}},$$
$$h_{yy} = h_{uu}\frac{y^2}{x^2+y^2} + h_u \frac{x^2}{(x^2+y^2)^{3/2}},$$
$$h_{xy} = h_{uu}\frac{xy}{x^2+y^2} - h_u \frac{xy}{(x^2+y^2)^{3/2}}.$$

Observe that due to homogeneity, the Hessian of $h(u,v)$ is zero. Hence

$$\begin{vmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{vmatrix} = \frac{1}{u} h_{uu} h_u,$$

and

$$\begin{vmatrix} h_{xx} & h_{xz} \\ h_{xz} & h_{zz} \end{vmatrix} + \begin{vmatrix} h_{yy} & h_{zy} \\ h_{zy} & h_{zz} \end{vmatrix} = \frac{1}{u} h_{vv} h_u.$$

□

Using the homogeneity of f_K and h_K we may extend the result above to the case $(x,y,z) \in \mathbb{R}^3$ (or, the same, $(u,v) \in \mathbb{R}^2$):

$$(u^2+v^2) f_K(u,v) = \frac{h_u}{u}\left(h_{uu} + h_{vv}\right). \tag{4}$$

Our goal is to solve this differential equation for h.

Lemma 2. *Define $h(\theta) = h(\cos\theta, \sin\theta)$ and $f(\theta) = f_K(\cos\theta, \sin\theta)$, $\theta \in [0, \pi/2]$. Then equation (4) can be rewritten as follows*

$$\cos\theta f(\theta) = \left(h + h''\right)\left(h\cos\theta - h'\sin\theta\right). \tag{5}$$

Proof : We pass to polar coordinates in R^2 and use the fact that

$$h_u = \cos\theta h_r - \frac{\sin\theta}{r} h_\theta,$$

and

$$h_{uu} + h_{vv} = \frac{1}{r} h_{rr} + \frac{1}{r} h_r + \frac{1}{r^2} h_{\theta\theta}.$$

Since h is a homogeneous function of degree one, we have

$$h_u = \cos\theta\, h(\theta) - \sin\theta\, h_\theta(\theta), \tag{6}$$

and

$$h_{uu} + h_{vv} = \frac{1}{r}\, h(\theta) + \frac{1}{r}\, h_{\theta\theta}(\theta).$$

Finally we plug these formulas into (4) and get

$$r^2 r^{-4} f_K(\theta) = \frac{1}{r\cos\theta}\left(\cos\theta h(\theta) - \sin\theta h_\theta(\theta)\right)\left(\frac{1}{r}h(\theta) + \frac{1}{r}h_{\theta\theta}(\theta)\right).$$

This gives the desired result.

□

Next we denote $\phi(\theta) = h\cos\theta - h'\sin\theta$ and observe that

$$\phi'(\theta) = h'\cos\theta - h\sin\theta - h''\sin\theta - h'\cos\theta = -\sin\theta(h + h'').$$

Equation (5) becomes

$$-\sin\theta\,\cos\theta f(\theta) = \phi(\theta)\phi'(\theta),$$

or

$$c_1 - \int\limits_0^\theta \sin(2\alpha)\, f(\alpha) d\alpha = \phi^2(\theta).$$

Lemma 3. *We have*

$$\phi(\theta) = \sqrt{\int\limits_\theta^{\frac{\pi}{2}} \sin(2\alpha)\, f(\alpha) d\alpha}.$$

Proof : Since $\phi(0) = h(0)$, we obtain

$$h^2(0) - \int\limits_0^\theta \sin(2\alpha)\, f(\alpha) d\alpha = \phi^2(\theta).$$

Now the Cauchy projection formula ([Sc], [Ga3]), and the fact that the projection of K onto the xy-plane is a disk of radius $h(0)$, give

$$\pi h^2(0) = \mathrm{Vol}_2(K|e_3^\perp) =$$

$$\frac{1}{2}\int\limits_{S^2} |z| f_K(x,y,z) d\sigma(x,y,z) = \pi \int\limits_0^{\pi/2} \sin(2\alpha)\, f(\alpha) d\alpha.$$

Thus

$$\phi^2(\theta) = \int\limits_{\theta}^{\pi/2} \sin(2\alpha)\, f(\alpha) d\alpha.$$

It remains to show that ϕ is nonnegative. This is a consequence of the fact that $\phi(\theta) = h_u(\theta)$ (see (6)), $\cos\theta \geq 0$ and the following proposition.

Proposition. *Let $L \subset R^n$ be of class C^1_+. Assume also that if $(x_1, \cdots, x_n) \in L$, then $(\varepsilon_1 x_1, \cdots \varepsilon_n x_n) \in L$ for any choice of signs $\varepsilon_1, \cdots, \varepsilon_n$. We have*

$$u_i \frac{\partial h_L}{\partial x_i}(u_1, \cdots, u_i, \cdots, u_n) \geq 0.$$

Proof: It is well known ([Sc], p. 40) that

$$\max_{y \in L} u \cdot y = u \cdot \mathrm{grad} h_L(u).$$

Assume that $u_i \frac{\partial h_L}{\partial x_i}(u) < 0$ for some $1 \leq i \leq n$. To get a contradiction we consider a point $y \in L$ such that all coordinates of y, with the exception of the ith, are equal to coordinates of $\mathrm{grad} h_L(u)$, and $y_i = -\frac{\partial h_L}{\partial x_i}(u)$. But then

$$u \cdot y > u \cdot \mathrm{grad} h_L(u).$$

□

To obtain a formula for h it remains to solve

$$\phi(\theta) = h \cos\theta - h' \sin\theta, \tag{7}$$

or after differentiation:

$$-\frac{\phi'(\theta)}{\sin\theta} = h'' + h. \tag{8}$$

Note that $\phi'(\theta)/\sin\theta$ is a continuous function on $[0, \pi/2]$. Using standard method we solve (8) with the initial values

$$h(0) = \sqrt{\int\limits_{0}^{\pi/2} \sin(2\alpha)\, f(\alpha) d\alpha}$$

and $h'(0) = 0$, (the last one comes from the fact that K has a smooth boundary, and $h(\theta)$ is an even function):

$$h(\theta) = \cos\theta \sqrt{\int\limits_{\theta}^{\pi/2} \sin(2\alpha)\, f(\alpha) d\alpha} + \sin\theta \int_0^{\theta} \frac{\cos^2 t f(t) dt}{\sqrt{\int\limits_{t}^{\pi/2} \sin(2\alpha)\, f(\alpha) d\alpha}}.$$

3. From projections to the curvature function

The volume of the projection of the convex body K is connected with the curvature function f_K via the formula of Cauchy:

$$\mathrm{Vol}_2\left(K\middle|u^\perp\right) = \frac{1}{2}\int\limits_{S^2} |u\cdot(x,y,z)| f_K(x,y,z)d\sigma(x,y,z). \tag{9}$$

Thus, in view of Section 2, it is enough to invert (9). Observe that in the case of a body of revolution the functions f_K and $u \to \mathrm{Vol}_2(K|u^\perp)$ are invariant under rotations around the axis, so it is enough to invert (9) at the point $u = (0, s, \sqrt{1-s^2})$, $s \in [0,1]$. Denote

$$\varphi(s) = \mathrm{Vol}_2(K|(0,s,\sqrt{1-s^2})^\perp).$$

Then (9) has the form

$$\varphi(s) = \int_{-1}^{1} f(z)dz \int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} \left|t\sqrt{1-z^2}s + z\sqrt{1-s^2}\right| dt.$$

We substitute $t\sqrt{1-z^2}s = \eta$, and use the evenness of $f(z)$ to get:

$$\varphi(s) = 2\int\limits_0^1 f(z)dz \int\limits_{-s\sqrt{1-z^2}}^{s\sqrt{1-z^2}} \left[(1-z^2)s^2-\eta^2\right]^{-\frac{1}{2}} \left|\eta + z\sqrt{1-s^2}\right| d\eta =$$

$$2\left(\int\limits_0^s + \int\limits_s^1\right) f(z)dz \int\limits_{-s\sqrt{1-z^2}}^{s\sqrt{1-z^2}} \left[(1-z^2)s^2-\eta^2\right]^{-\frac{1}{2}} \left|\eta + z\sqrt{1-s^2}\right| d\eta =$$

$$= I_1 + I_2.$$

Observe that $z < s$ implies $z\sqrt{1-s^2} < s\sqrt{1-z^2}$, so

$$I_1 = 2\Big[\int\limits_0^s f(z)dz \int\limits_{-z\sqrt{1-s^2}}^{s\sqrt{1-z^2}} \left[(1-z^2)s^2-\eta^2\right]^{-\frac{1}{2}} \left(\eta + z\sqrt{1-s^2}\right) d\eta +$$

$$\int\limits_0^s f(z)dz \int\limits_{-s\sqrt{1-z^2}}^{-z\sqrt{1-s^2}} \left[(1-z^2)s^2-\eta^2\right]^{-\frac{1}{2}} \left(-\eta - z\sqrt{1-s^2}\right) d\eta\Big].$$

This gives

$$I_1 = 2\Big[\int\limits_0^s z\sqrt{1-s^2} f(z)dz \int\limits_{-s\sqrt{1-z^2}}^{s\sqrt{1-z^2}} \left[(1-z^2)s^2-\eta^2\right]^{-\frac{1}{2}} d\eta -$$

$$-2\int\limits_0^s f(z)dz \int\limits_{-s\sqrt{1-z^2}}^{-z\sqrt{1-s^2}} \left[(1-z^2)s^2-\eta^2\right]^{-\frac{1}{2}} \eta d\eta -$$

$$-2\int\limits_0^s z\sqrt{1-s^2}f(z)dz \int\limits_{-s\sqrt{1-z^2}}^{-z\sqrt{1-s^2}} \left[(1-z^2)s^2-\eta^2\right]^{-\frac{1}{2}} d\eta\Big].$$

Similarly, $1 \geq z \geq s$ implies $s\sqrt{1-z^2} \leq z\sqrt{1-s^2}$, so

$$I_2 = 2\int\limits_s^1 f(z)z\sqrt{1-s^2}dz \int\limits_{-s\sqrt{1-z^2}}^{s\sqrt{1-z^2}} \left[(1-z^2)s^2-\eta^2\right]^{-\frac{1}{2}} d\eta.$$

Now we have

$$\varphi(s) = I_1 + I_2 = 2\Big[\int\limits_0^1 z\sqrt{1-s^2}f(z)dz \int\limits_{-s\sqrt{1-z^2}}^{s\sqrt{1-z^2}} \left[(1-z^2)s^2-\eta^2\right]^{-\frac{1}{2}} d\eta -$$

$$-2\int\limits_0^s f(z)dz \int\limits_{-s\sqrt{1-z^2}}^{-z\sqrt{1-s^2}} \left[(1-z^2)s^2-\eta^2\right]^{-\frac{1}{2}} \eta d\eta -$$

$$-2\int\limits_0^s z\sqrt{1-s^2}f(z)dz \int\limits_{-s\sqrt{1-z^2}}^{-z\sqrt{1-s^2}} \left[(1-z^2)s^2-\eta^2\right]^{-\frac{1}{2}} d\eta\Big].$$

In other words,

$$\varphi(s) = 4\Big[\frac{\pi}{4}\sqrt{1-s^2}\int\limits_0^1 zf(z)dz + \int_0^s \sqrt{s^2-z^2}f(z)dz -$$

$$-\int_0^s z\sqrt{1-s^2}f(z)\arccos\frac{z\sqrt{1-s^2}}{s\sqrt{1-z^2}}dz\Big].$$

We can simplify this formula, dividing both sides by $\sqrt{1-s^2}$ and taking the derivative with respect to $s \in (0,1)$. We get

$$\frac{d}{ds}\left(\frac{\varphi(s)}{\sqrt{1-s^2}}\right) = \frac{4}{s(1-s^2)^{3/2}}\int_0^s \sqrt{s^2-z^2}f(z)dz.$$

We define

$$g(s) = \frac{s(1-s^2)^{3/2}}{4}\frac{d}{ds}\left(\frac{\varphi(s)}{\sqrt{1-s^2}}\right),$$

to get the integral equation

$$g(s) = \int_0^s \sqrt{s^2 - z^2} f(z) dz,$$

which can be inverted by standard methods, see for example ([H], p. 11, $n = 4$):

$$f(z) = \frac{2}{3} \int\limits_0^z z(z^2 - s^2)^{-\frac{1}{2}} \left(\frac{g'(s)}{s} \right)' ds, \qquad 0 \le z \le 1.$$

Finally

$$f(z) = \frac{1}{6} \int\limits_0^z z(z^2 - s^2)^{-\frac{1}{2}} \frac{d}{ds} \left(\frac{1}{s} \frac{d}{ds} \left(s(1 - s^2)^{3/2} \frac{d}{ds} \left(\frac{\varphi(s)}{\sqrt{1 - s^2}} \right) \right) \right) ds.$$

$\square$

The following result is well-known (see [Sc], p. 431, relation (A.16) and [See]). We include it here for the convenience of the reader.

Lemma 4. *The function*

$$\mathrm{Vol}_2(K|\cdot^{\perp}) : u \in S^2 \to \mathrm{Vol}_2(K|u^{\perp})$$

belongs to $C^3(S^2)$*, provided* $f_K \in C^3(S^2)$*.*

Proof : Let $n \ge 3$, $d_n(m) = \frac{(n+2m-2)(n+m-3)!}{m!(n-2)!}$, and let

$$\sum_{m=1}^{\infty} \sum_{\mu=1}^{d_n(m)} f_{m,\mu} Y_{m,\mu}(\theta), \qquad f_{m,\mu} = \int\limits_{S^{n-1}} f(\sigma) Y_{m,\mu}(\sigma) d\sigma,$$

be the Fourier-Laplace series of $f \in L^2(S^{n-1})$. It is well-known (see [See]), that $f \in C^{2r}(S^{n-1})$ implies

$$|f_{m,\mu}| \le c\, m^{-2r}, \qquad |D^j Y_m(\theta)| \le c\, m^{|j| + \frac{n-2}{2}},$$

for all multi-indices j, $|j| = j_1 + ... + j_n = 0, 1, 2, ...,$

$$D^j = \frac{\partial^{|j|}}{(\partial x_1)^{j_1} ... (\partial x_n)^{j_n}}.$$

Let $2r > 3(n-2)/2 + |j| + 1$. It follows that the derivatives D^j of the Fourier-Laplace series of f converge uniformly and absolutely to $D^j f(\theta)$. Indeed, $d_n(m) \sim c\, m^{n-2}$ as $m \to \infty$, and we have

$$|\sum_{m=1}^{\infty} \sum_{\mu=1}^{d_n(m)} f_{m,\mu} D^j Y_{m,\mu}(\theta)| \le C \sum_{m=1}^{\infty} \frac{1}{m^{2r - 3/2(n-2) - |j|}} < \infty.$$

Let $f_K \in C^{2r}(S^{n-1})$, $2r > |j| + n - 7/2$. Then the Cauchy formula and the Fourier-Laplace decomposition of the Cosine transform imply

$$D^j \mathrm{Vol}_{n-1}(K|\theta^\perp) = D^j \Big(\frac{1}{2} \int\limits_{S^{n-1}} |\theta \cdot x| f_K(x) dx \Big) =$$

$$= \sum_{m=1}^{\infty} \sum_{\mu=1}^{d_n(m)} \gamma_m \, (f_K)_{m,\mu} \, D^j Y_{m,\mu}(\theta),$$

where $\gamma_m \sim c\, m^{-(3+n)/2}$ as $m \to \infty$. Moreover,

$$|\sum_{m=1}^{\infty} \sum_{\mu=1}^{d_n(m)} \gamma_m \, (f_K)_{m,\mu} \, D^j Y_{m,\mu}(\theta)| \le C \sum_{m=1}^{\infty} \frac{1}{m^{2r-3/2(n-2)-|j|+(3+n)/2}} < \infty.$$

Thus, the function $\mathrm{Vol}_2(K|\cdot^\perp) : u \in S^2 \to \mathrm{Vol}_2(K|u^\perp)$ is three times continuously differentiable, provided $f_K \in C^{2r}(S^2)$, $2r \ge 3$.

$\square$

Acknowledgments: We wish to thank Alexander Koldobsky for fruitful discussions and for a series of suggestions that improved the paper. We thank Alex Iosevich for pointing to the book of A. Pogorelov [Pog].

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