Gaussian Measure of Sections of convex bodies

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Abstract

In this paper we study properties of sections of convex bodies with respect to the Gaussian measure. We develop a formula connecting the Minkowski functional of a convex symmetric body K with the Gaussian measure of its sections. Using this formula we solve an analog of the Busemann-Petty problem for Gaussian measures.

Key words: Convex body, Gaussian Measure, Busemann-Petty problem

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1 Introduction

The standard Gaussian measure on $\mathbb{R}^n$ is given by its density:

$$\gamma_n(K) = \frac{1}{(\sqrt{2\pi})^n} \int\limits_K e^{-\frac{|x|^2}{2}} dx,$$

where $|x|$ denotes the ℓ_2 norm on $\mathbb{R}^n$. Let $\mathcal{C}_n$ denote the collection of convex closed subsets of $\mathbb{R}^n$ with nonempty interior, which are symmetric about the origin. In this paper we give an answer to the following question:
Gaussian Busemann-Petty problem (GBP): *Assume* K,$L \in \mathcal{C}_n$ *and*

$$\gamma_{n-1}(K \cap \xi^\perp) \le \gamma_{n-1}(L \cap \xi^\perp), \ \ \forall \xi \in S^{n-1},$$

where $K \cap \xi^\perp$ *denotes the section of* K *by the central hyperplane orthogonal to* ξ. *Does it follow that*

$$\gamma_n(K) \le \gamma_n(L)?$$

In Section 3 we show that the answer is affirmative if $n \le 4$ and it is negative if $n \ge 5$.

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This problem is a Gaussian analog of the Busemann-Petty problem, posed in 1956 (see [3]) and asking the same question with Lebesgue measure in place of the Gaussian measure. The answer is affirmative if $n \leq 4$ and negative if $n \geq 5$, The solution appeared as the result of a sequence of papers: [16] $n \geq 12$, [1] $n \geq 10$, [9] and [2] $n \geq 7$, [18] and [4] $n \geq 5$, [5] $n = 3$, [22] and [7] $n = 4$ (we refer to [22], [7] and [14] for more historical details).

Let us outline the analytic solution to the Busemann-Petty problem from [7]. The first ingredient is a connection with intersection bodies found by Lutwak [17]. Let K and M be symmetric star bodies in $\mathbb{R}^n$. We say that K is the *intersection body of* M if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the central hyperplane section of L perpendicular to this direction, i.e. for every $\xi \in S^{n-1}$, $\|\xi\|_K^{-1} = \mathrm{Vol}(M \cap \xi^\perp)$. A more general class of *intersection bodies* can be defined as the closure of the class of intersection bodies of star bodies in the radial metric .

Lutwak [17] proved that if K is an intersection body then the answer to the Busemann-Petty problem is affirmative for every L, and, on the other hand, if L not an intersection body, then one can perturb it to construct a body K giving together with L a counterexample. Therefore, the answer to the Busemann-Petty problem in $\mathbb{R}^n$ is affirmative if and only if every symmetric convex body in $\mathbb{R}^n$ is an intersection body.

The second ingredient is the following Fourier analytic characterization of intersection bodies found by Koldobsky [12]: an origin symmetric star body K in $\mathbb{R}^n$ is an intersection body if and only if the function $\|\cdot\|_K^{-1}$ represents a positive definite distribution on $\mathbb{R}^n$. The characterization is based on the following formula ([11], Theorem 1)

$$\frac{1}{\pi(n-1)}(\|x\|_K^{-n+1})^\wedge(\xi) = A_{K,\xi}(0), \tag{1}$$

where $A_{K,\xi}(t) = \mathrm{Vol}(K \cap \{\xi^\perp + t\xi\})$.

The formula (1) was generalized by Gardner, Koldobsky and Schlumprecht [7] to

(i) If k is even

$$(\|\cdot\|_K^{-n+k+1})^\wedge(\xi) = (-1)^{k/2}\pi(n-k-1)A_{K,\xi}^{(k)}(0); \tag{2}$$

(ii) If k is odd

$$(\|\cdot\|_K^{-n+k+1})^\wedge(\xi) = c_{n,k}\int_0^\infty \frac{A_{K,\xi}(z) - A_{K,\xi}(0) - \ldots - A_{K,\xi}^{(k-1)}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}}dz, \tag{3}$$

where $c_{n,k} = (-1)^{(k+1)/2} 2(n-k-1)k!$ and $A^{(k)}_{K,\xi}$ stands for the derivative of the order k of the function $A_{K,\xi}$.

Finally, if the body K is convex then, by Brunn's theorem, the central section is maximal among all sections orthogonal to a given direction, so $A''_{K,\xi}(0) \leq 0$ for every $\xi \in S^{n-1}$, and, therefore, putting $n = 4$ in (2) and applying Fourier analytic characterization of intersection bodies, we get that every symmetric convex body in $\mathbb{R}^4$ is an intersection body. However, if $n = 5$ we have to deal with the third derivative of $A_{K,\xi}$ which is not controlled by convexity, and one construct symmetric convex bodies in $\mathbb{R}^5$ that are not intersection bodies (see [7]).

In the first section of this paper we introduce a Fourier analytic formula for $\gamma_{n-1}(K \cap \xi^{\perp})$ (see Theorem 1 below). Using this formula and the version of Parseval's identity on the sphere ([13], Lemma 3) we prove that intersection bodies play the same role in GBP as in the original Busemann-Petty problem. In Theorems 3, 4 below we show that if $\| \cdot \|_K^{-1}$ represents a positive definite distribution on $\mathbb{R}^n$, then the answer to GBP is affirmative. Finally Theorem 5 below shows that if L is not an intersection body then one can construct a counterexample to GBP. This, together with results about intersection bodies [7], gives affirmative answer to GBP in dimensions 3 and 4 and negative answer in dimension $n \geq 5$.

Some additional remarks are provided in the last section. We show that a counterexample to GBP problem in $\mathbb{R}^n$, $n \geq 5$ can be also constructed directly from the examples to the original Busemann-Petty problem. We also show that this method can be used to give an answer to the projection counterpart of GBP.

2 The main formula

In this paper we operate with the Fourier transform of distributions (see [8] for exact definitions and properties). We denote by $\mathcal{S}$ the space of rapidly decreasing infinitely differentiable functions (test functions) on $\mathbb{R}^n$ with values in $\mathbb{C}$. By $\mathcal{S}'$ we denote the space of distributions over $\mathcal{S}$. The Fourier transform of a distribution f is defined by $\langle \hat{f}, \hat{\phi} \rangle = (2\pi)^n \langle f, \phi \rangle$, for every test function ϕ.

The *spherical Radon transform* is the bounded linear operator on $C(S^{n-1})$ defined by

$$\mathcal{R}f(\xi) = \int\limits_{S^{n-1} \cap \xi^{\perp}} f(x)dx, \;\; f \in C(S^{n-1}), \;\; \xi \in S^{n-1}.$$

Koldobsky ([10], Lemma 4) proved that if $f(x)$ is an even homogeneous function of degree $-n+1$ on $\mathbb{R}^n \setminus \{0\}$, $n > 1$ so that $f|_{S^{n-1}} \in L_1(S^{n-1})$, then

$$\mathcal{R}f(\xi) = \frac{1}{\pi}\hat{f}(\xi), \qquad \forall \xi \in S^{n-1}. \tag{4}$$

Let K be an origin-symmetric star-shaped body. We denote by

$$||x||_K = \min\{\alpha > 0 \, : \, x \in \alpha K\}$$

the Minkowski functional on $\mathbb{R}^n$ generated by K.

Theorem 1 *Let K be a symmetric star-shaped body in $\mathbb{R}^n$, then*

$$\gamma_{n-1}(K \cap \xi^\perp) = \frac{1}{(\sqrt{2\pi})^{n-1}\pi} \left(|x|^{-n+1} \int\limits_0^{|x|/||x||_K} t^{n-2} e^{-\frac{t^2}{2}} dt \right)^\wedge (\xi).$$

Proof : If χ is the indicator function of the interval $[-1, 1]$ then, passing to polar coordinates in the hyperplane $\xi^\perp$ we get

$$(\sqrt{2\pi})^{n-1}\gamma_{n-1}(K \cap \xi^\perp) = \int\limits_{(x,\xi)=0} \chi(\|x\|_K) e^{-\frac{|x|^2}{2}} dx = \int\limits_{S^{n-1}\cap\xi^\perp} \int\limits_0^{||\theta||_K^{-1}} t^{n-2} e^{-\frac{t^2}{2}} dt d\theta.$$

We extend the function under the spherical integral to a homogeneous of degree $-n+1$ function on $\mathbb{R}^n$ and apply (4) to get

$$= \int\limits_{S^{n-1}\cap\xi^\perp} |x|^{-n+1} \int\limits_0^{\frac{|x|}{||x||_K}} t^{n-2} e^{-\frac{t^2}{2}} dt dx = \mathcal{R} \left(|x|^{-n+1} \int\limits_0^{\frac{|x|}{||x||_K}} t^{n-2} e^{-\frac{t^2}{2}} dt \right)(\xi)$$

$$= \frac{1}{\pi} \left(|x|^{-n+1} \int\limits_0^{|x|/||x||_K} t^{n-2} e^{-\frac{t^2}{2}} dt \right)^\wedge (\xi). \tag{5}$$

□

Remark 1: Note that the function in (5) is homogeneous of degree -1 (with respect to $\xi \in \mathbb{R}^n$). This gives a natural extension of $G_K(\xi) = \gamma_{n-1}(K \cap \xi^\perp)$ to a homogeneous function of degree -1, i.e.

$$G_K(\nu) = |\nu|^{-1} G_K(\nu/|\nu|) = |\nu|^{-1}\gamma_{n-1}(K \cap (\nu/|\nu|)^\perp),$$

for $\nu \in \mathbb{R}^n$.

Remark 2: If $n = 3$ then

$$\gamma_2(K \cap \xi^\perp) = 1 - \frac{1}{2\pi^2}\left(|x|^{-2}e^{-\frac{|x|^2}{2||x||_K^2}}\right)^\wedge(\xi).$$

Indeed, from Theorem 1 and the well-known formula for the Fourier transform of powers of the Euclidean norm (see [8], p. 194) we get

$$\gamma_2(K \cap \xi^\perp) = \frac{1}{2\pi^2}\widehat{|x|^{-2}}(\xi) - \frac{1}{2\pi^2}\left(|x|^{-2}e^{-\frac{|x|^2}{2||x||_K^2}}\right)^\wedge(\xi)$$

$$= 1 - \frac{1}{2\pi^2}\left(|x|^{-2}e^{-\frac{|x|^2}{2||x||_K^2}}\right)^\wedge(\xi).$$

Theorem 1 also implies that a symmetric body is uniquely determined by the Gaussian measure of its sections:

Theorem 2 *Let K and L be star-shaped origin symmetric bodies in $\mathbb{R}^n$. If*

$$\gamma_{n-1}(K \cap \xi^\perp) = \gamma_{n-1}(L \cap \xi^\perp),\ \forall \xi \in S^{n-1},$$

then $K = L$.

Proof : Remark 1 after Theorem 1 gives that

$$G_K(\nu) = G_L(\nu),\ \ \forall \nu \in \mathbb{R}^n.$$

Applying the inverse Fourier transform to both sides of the latter equation and using Theorem 1 we get:

$$|x|^{-n+1}\int_0^{|x|/||x||_K} t^{n-2}e^{-\frac{t^2}{2}}dt = |x|^{-n+1}\int_0^{|x|/||x||_L} t^{n-2}e^{-\frac{t^2}{2}}dt,\ \ \forall x \in \mathbb{R}^n,$$

which gives

$$\int_0^{||x||_K^{-1}} t^{n-2}e^{-\frac{t^2}{2}}dt = \int_0^{||x||_L^{-1}} t^{n-2}e^{-\frac{t^2}{2}}dt,\ \ \forall x \in S^{n-1},$$

so $||x||_K = ||x||_L$.

$\square$

3 The Busemann-Petty Problem for Gaussian Measure

We start with positive answer to GBP in dimension 3.

Theorem 3 *Let K, $L \in \mathcal{C}_3$. Assume that*

$$\gamma_2(K \cap \xi^\perp) \le \gamma_2(L \cap \xi^\perp), \ \ \forall \xi \in S^2, \tag{6}$$

then

$$\gamma_3(K) \le \gamma_3(L).$$

Proof : We say that a body $K \subset \mathbb{R}^n$ is infinitely smooth if the restriction of the Minkowski functional to the sphere S^{n-1} belongs to the space $C^\infty(S^{n-1})$.

By elementary approximation, it is enough to prove the theorem in the case of infinitely smooth, symmetric, convex bodies K and L. Note that then ([7], Theorem 1 or [13], Remark 1) $\widehat{||\cdot||_K^{-1}}$ and $\widehat{||\cdot||_L^{-1}}$ are continuous functions on S^2, and the same argument can be applied to show that

$$\left(|x|^{-2} e^{-\frac{|x|^2}{2||x||_K^2}}\right)^\wedge \text{ and } \left(|x|^{-2} e^{-\frac{|x|^2}{2||x||_L^2}}\right)^\wedge$$

are also continuous functions on S^2.

From (6) and Remark 2 after Theorem 1,

$$-\left(|x|^{-2} e^{-\frac{|x|^2}{2||x||_K^2}}\right)^\wedge (\xi) \le -\left(|x|^{-2} e^{-\frac{|x|^2}{2||x||_L^2}}\right)^\wedge (\xi), \qquad \forall \xi \in S^2.$$

Every convex symmetric body in $\mathbb{R}^3$ is an intersection body ([5], [7]), so $\widehat{||\cdot||_K^{-1}} \ge 0$ ([12], Theorem 1) and

$$-\widehat{||\cdot||_K^{-1}} \left(|x|^{-2} e^{-\frac{|x|^2}{2||x||_K^2}}\right)^\wedge (\xi) \le -\widehat{||\cdot||_K^{-1}} \left(|x|^{-2} e^{-\frac{|x|^2}{2||x||_L^2}}\right)^\wedge (\xi), \ \forall \xi \in S^2.$$

In particular,

$$-\int_{S^2} \widehat{||\cdot||_K^{-1}} \left(|x|^{-2} e^{-\frac{|x|^2}{2||x||_K^2}}\right)^\wedge (\xi) d\xi \le -\int_{S^2} \widehat{||\cdot||_K^{-1}} \left(|x|^{-2} e^{-\frac{|x|^2}{2||x||_L^2}}\right)^\wedge (\xi) d\xi.$$

Note that the sum of degrees of homogeneity of functions under both integrals is -3, so we may apply a version of the Parseval identity on S^2 ([13], Lemma 4) to get :

$$-\int_{S^2} ||x||_K^{-1} e^{-\frac{1}{2||x||_K^2}} dx \le -\int_{S^2} ||x||_K^{-1} e^{-\frac{1}{2||x||_L^2}} dx. \tag{7}$$

Passing to polar coordinates and integrating by parts we get the following expression for $\gamma_3(K)$:

$$\gamma_3(K) = -\frac{1}{(\sqrt{2\pi})^3} \int\limits_{S^2} ||x||_K^{-1} e^{-\frac{1}{2||x||_K^2}} dx + \frac{1}{(\sqrt{2\pi})^3} \int\limits_{S^2} \int\limits_0^{||x||_K^{-1}} e^{-\frac{t^2}{2}} dt dx.$$

Then, to show that $\gamma_3(K) \leq \gamma_3(L)$ it is enough to prove

$$-\int\limits_{S^2} ||x||_K^{-1} e^{-\frac{1}{2||x||_K^2}} dx \leq -\int\limits_{S^2} ||x||_L^{-1} e^{-\frac{1}{2||x||_L^2}} dx + \int\limits_{S^2} \int\limits_{||x||_K^{-1}}^{||x||_L^{-1}} e^{-\frac{t^2}{2}} dt dx. \tag{8}$$

We may apply the inequality (7) to claim that (8) follows from

$$-\int\limits_{S^2} ||x||_K^{-1} e^{-\frac{1}{2||x||_L^2}} dx \leq -\int\limits_{S^2} ||x||_L^{-1} e^{-\frac{1}{2||x||_L^2}} dx + \int\limits_{S^2} \int\limits_{||x||_K^{-1}}^{||x||_L^{-1}} e^{-\frac{t^2}{2}} dt dx.$$

If we denote $||x||_K^{-1} = a$, $||x||_L^{-1} = b$ the latter inequality follows from

$$(b-a)e^{-b^2/2} \leq \int\limits_a^b e^{-\frac{t^2}{2}} dt, \quad \forall\, a, b \geq 0.$$

$\square$

Theorem 4 (General Case) *Consider origin symmetric convex bodies K and L in $\mathbb{R}^n$, $n \geq 4$, and suppose that K be an intersection body. Then from*

$$\gamma_{n-1}(K \cap \xi^\perp) \leq \gamma_{n-1}(L \cap \xi^\perp), \;\; \forall \xi \in S^{n-1}, \tag{9}$$

it follows that

$$\gamma_n(K) \leq \gamma_n(L).$$

Proof : As in the proof of Theorem 6, it is enough to prove Theorem 4 in the case of infinitely smooth, bodies K and L. Then, again, $\widehat{||\cdot||_K^{-1}}, \widehat{||\cdot||_L^{-1}}$, and

$$\left(|x|^{-n+1} \int\limits_0^{|x|/||x||_K} t^{n-2} e^{-\frac{t^2}{2}} dt\right)^\wedge, \;\; \left(|x|^{-n+1} \int\limits_0^{|x|/||x||_L} t^{n-2} e^{-\frac{t^2}{2}} dt\right)^\wedge$$

are continuous functions on S^{n-1}.

We apply Theorem 1 to rewrite (9) as

$$\left(|x|^{-n+1}\int\limits_0^{|x|/||x||_K} t^{n-2}e^{-\frac{t^2}{2}}dt\right)^{\wedge}(\xi) \le \left(|x|^{-n+1}\int\limits_0^{|x|/||x||_L} t^{n-2}e^{-\frac{t^2}{2}}dt\right)^{\wedge}(\xi).$$

Multiplying both sides of the above equation by $\widehat{||\cdot||_K^{-1}}(\xi)$, integrating over S^{n-1} and applying Parseval's identity in the form of [13], Lemma 4 (note the sum of degrees of homogeneity is $-n$) we get:

$$\int\limits_{S^{n-1}} ||x||_K^{-1}\left(\int\limits_0^{||x||_K^{-1}} t^{n-2}e^{-\frac{t^2}{2}}dt\right)dx \le \int\limits_{S^{n-1}} ||x||_K^{-1}\left(\int\limits_0^{||x||_L^{-1}} t^{n-2}e^{-\frac{t^2}{2}}dt\right)dx.$$

Next, integration by parts gives:

$$\begin{aligned}-\int\limits_{S^{n-1}} ||x||_K^{-n+2}e^{-\frac{1}{2||x||_K^2}}dx \le -\int\limits_{S^{n-1}} & ||x||_K^{-1}||x||_L^{-n+3}e^{-\frac{1}{2||x||_L^2}} \\ & +(n-3)||x||_K^{-1}\left(\int\limits_{||x||_K^{-1}}^{||x||_L^{-1}} t^{n-4}e^{-\frac{t^2}{2}}dt\right)dx.\end{aligned} \tag{10}$$

Passing to polar coordinates in $\mathbb{R}^n$ and integrating by parts we get that $\gamma_n(K) \le \gamma_n(L)$ is equivalent to

$$-\int\limits_{S^{n-1}} ||x||_K^{-n+2}e^{-\frac{1}{2||x||_K^2}}dx$$

$$\le -\int\limits_{S^{n-1}} ||x||_L^{-n+2}e^{-\frac{1}{2||x||_L^2}}dx + (n-2)\int\limits_{S^{n-1}}\int\limits_{||x||_K^{-1}}^{||x||_L^{-1}} t^{n-3}e^{-\frac{t^2}{2}}dtdx.$$

Note that by (10), the latter inequality will follow from

$$-\int\limits_{S^{n-1}} ||x||_K^{-1}||x||_L^{-n+3}e^{-\frac{1}{2||x||_L^2}} + (n-3)||x||_K^{-1}\left(\int\limits_{||x||_K^{-1}}^{||x||_L^{-1}} t^{n-4}e^{-\frac{t^2}{2}}dt\right)dx$$

$$\le -\int\limits_{S^{n-1}} ||x||_L^{-n+2}e^{-\frac{1}{2||x||_L^2}}dx + (n-2)\int\limits_{S^{n-1}}\int\limits_{||x||_K^{-1}}^{||x||_L^{-1}} t^{n-3}e^{-\frac{t^2}{2}}dtdx.$$

Denote $||x||_K^{-1} = a$, $||x||_L^{-1} = b$, then the latter inequality follows from an elementary inequality for $a, b \geq 0$:

$$b^{n-3}(b-a)e^{-\frac{b^2}{2}} + (n-3)a\int_a^b t^{n-4}e^{-\frac{t^2}{2}}dt \leq (n-2)\int_a^b t^{n-3}e^{-\frac{t^2}{2}}dt. \qquad (11)$$

To verify (11) we fix $a > 0$ and define a function $h(b)$:

$$h(b) = (n-2)\int_a^b t^{n-3}e^{-\frac{t^2}{2}}dt - (b^{n-2} - b^{n-3}a)e^{-b^2/2} - (n-3)a\int_a^b t^{n-4}e^{-\frac{t^2}{2}}dt.$$

Then $h'(b) = e^{-b^2/2}b^{n-2}(b-a)$ and h attains its minimum at $b = a$.

$\square$

The affirmative solution to GBP in dimension 4 follows directly from Theorem 4 and the fact that every symmetric convex body K in $\mathbb{R}^4$ is an intersection body (see [7], [22]).

Corollary 1 *Let* $K, L \in \mathcal{C}_4$ *and assume that*

$$\gamma_3(K \cap \xi^\perp) \leq \gamma_3(L \cap \xi^\perp),$$

then

$$\gamma_4(K) \leq \gamma_4(L).$$

Next we give a negative answer to GBP in dimensions $n \geq 5$.

Theorem 5 *If* L *is an infinitely smooth, origin symmetric, convex body in* $\mathbb{R}^n$ *with positive curvature, and* $|| \cdot ||_L^{-1}$ *is not a positive definite distribution, then there exists a convex body* D *in* $\mathbb{R}^n$ *such that*

$$\gamma_{n-1}(D \cap \xi^\perp) \leq \gamma_{n-1}(L \cap \xi^\perp), \ \ \forall \xi \in S^{n-1},$$

but

$$\gamma_n(D) > \gamma_n(L).$$

Proof : Note that $\widehat{|| \cdot ||_L^{-1}}$ is a continuous function on S^{n-1} (see [7], Theorem 1 or [13], Remark 1). By our assumption, this function is negative on some open symmetric subset Ω of S^{n-1}. Let $f \in C^\infty(S^{n-1})$ be any non-negative even function supported in Ω. Extend f to a homogeneous function $f(\theta)r^{-1}$ of degree -1 on $\mathbb{R}^n$. Then the Fourier transform of f is a homogeneous function of degree $-n+1$: $\widehat{f(\theta)r^{-1}} = g(\theta)r^{-n+1}$, where g is an infinitely smooth function on S^{n-1} ([13], Lemma 5). Choosing a small $\varepsilon > 0$, we define a body D by

$$|x|^{-n+1} \int\limits_0^{|x|/||x||_D} t^{n-2} e^{-\frac{t^2}{2}} dt = |x|^{-n+1} \int\limits_0^{|x|/||x||_L} t^{n-2} e^{-\frac{t^2}{2}} dt - \varepsilon g(x/|x|)|x|^{-n+1}.$$

Then, one can choose a small enough ε so that the body D is convex. Indeed, for small enough ε we may define a function $\alpha_\varepsilon(x)$ such that

$$\int\limits_0^{|x|/||x||_L} t^{n-2} e^{-\frac{t^2}{2}} dt - \varepsilon g(x/|x|) = \int\limits_0^{|x|/||x||_L - \alpha_\varepsilon(x/|x|)} t^{n-2} e^{-\frac{t^2}{2}} dt,$$

then

$$\frac{|x|}{||x||_D} = \frac{|x|}{||x||_L} - \alpha_\varepsilon(x/|x|),$$

or

$$||x||_D^{-1} = ||x||_L^{-1} - \alpha_\varepsilon(x/|x|)|x|^{-1}.$$

Moreover $\alpha_\varepsilon(\theta)$ and its derivatives converge uniformly to 0 (for $\theta \in S^{n-1}$). Using that L is convex with positive curvature, one can choose a small enough ε so that the body D is convex.

Now we are ready to finish the proof:

$$\gamma_{n-1}(D \cap \xi^\perp) = \frac{1}{(\sqrt{2\pi})^{n-1}\pi} \left(|x|^{-n+1} \int\limits_0^{|x|/||x||_D} t^{n-2} e^{-\frac{t^2}{2}} dt \right)^\wedge (\xi)$$

$$= \frac{1}{(\sqrt{2\pi})^{n-1}\pi} \left(|x|^{-n+1} \int\limits_0^{|x|/||x||_L} t^{n-2} e^{-\frac{t^2}{2}} dt \right)^\wedge (\xi) - \frac{(2\pi)^n \varepsilon f(\xi)}{(\sqrt{2\pi})^{n-1}\pi}$$

$$\leq \gamma_{n-1}(L \cap \xi^\perp).$$

On the other hand, the function f is positive only where $\widehat{||x||_L^{-1}}$ is negative so

$$\widehat{||x||_L^{-1}} \left(|x|^{-n+1} \int\limits_0^{|x|/||x||_D} t^{n-2} e^{-\frac{t^2}{2}} dt \right)^\wedge (\xi)$$

$$= \widehat{||x||_L^{-1}} \left(|x|^{-n+1} \int\limits_0^{|x|/||x||_L} t^{n-2} e^{-\frac{t^2}{2}} dt \right)^\wedge (\xi) - (2\pi)^n \widehat{||x||_L^{-1}}(\xi) \varepsilon f(\xi)$$

$$\geq \widehat{||x||_L^{-1}} \left(|x|^{-n+1} \int\limits_0^{|x|/||x||_L} t^{n-2} e^{-\frac{t^2}{2}} dt \right)^\wedge (\xi).$$

Now the same computations as in the previous theorem give

$$\gamma_n(D) > \gamma_n(L).$$

$\square$

It was proved in [7] (see also [14]) that, for each fixed $n \geq 5$, there exists an infinitely smooth, symmetric, convex body $L \subset \mathbb{R}^n$, so that $||\cdot||_L^{-1}$ is not positive definite. Therefore,

Corollary 2 *For any $n \geq 5$ there exist convex symmetric bodies D and L in $\mathbb{R}^n$ such that*

$$\gamma_{n-1}(D \cap \xi^\perp) \leq \gamma_{n-1}(L \cap \xi^\perp), \ \forall \xi \in S^{n-1},$$

but

$$\gamma_n(D) > \gamma_n(L).$$

4 Remarks

Remark 1: One can present a different proof of Corollary 2, based on the original counterexample for the Busemann-Petty problem and the "flatness" of the Gaussian density near the origin. Note that $e^{-|x|^2/2} \leq 1$ for any $x \in \mathbb{R}^n$ and $e^{-|x|^2/2} \geq e^{-\delta^2/2}$ for any $x \in \delta B_2$. Then, for any $K \in \delta B_2$:

$$\mathrm{Vol}(K) = \int\limits_K dx \geq \int\limits_K e^{-\frac{|x|^2}{2}} dx = (\sqrt{2\pi})^n \gamma_n(K) \geq \int\limits_K e^{-\frac{\delta^2}{2}} dx = e^{-\frac{\delta^2}{2}} \mathrm{Vol}(K),$$

so

$$1 \geq \frac{(\sqrt{2\pi})^n \gamma_n(K)}{\mathrm{Vol}(K)} \geq e^{-\frac{\delta^2}{2}} \tag{12}$$

and

$$1 \geq \frac{(\sqrt{2\pi})^{n-1} \gamma_{n-1}(K \cap \xi^\perp)}{\mathrm{Vol}(K \cap \xi^\perp)} \geq e^{-\frac{\delta^2}{2}}, \ \ \forall \xi \in S^{n-1}. \tag{13}$$

Now, for $n \geq 5$, we may consider convex, origin symmetric sets K and L in $\mathbb{R}^n$, which give a counterexample to the Busemann-Petty problem. Clearly, any dilations αK and αL will also provide a counterexample. Consider a function $\alpha(\delta) = \max\{\alpha : \alpha K, \alpha L \subset \delta B_2\}$. Note that if $\mathrm{Vol}(L) < \mathrm{Vol}(K)$ then there is a δ_0 such that for all $\delta < \delta_0$:

$$\mathrm{Vol}(L) < e^{-\frac{\delta^2}{2}} \mathrm{Vol}(K).$$

Then for $\delta < \delta_0$:

$$\gamma_n(\alpha(\delta)L) \leq \frac{1}{(\sqrt{2\pi})^n} \mathrm{Vol}(\alpha(\delta)L) < e^{-\frac{\delta^2}{2}} \frac{1}{(\sqrt{2\pi})^n} \mathrm{Vol}(\alpha(\delta)K) \leq \gamma_n(\alpha(\delta)K).$$

Using that K and L are smooth convex bodies and

$$\mathrm{Vol}(K \cap \xi^{\perp}) < \mathrm{Vol}(L \cap \xi^{\perp}), \;\; \forall \xi \in S^{n-1},$$

we may choose a δ_1 such that for any $\delta < \delta_1$:

$$\mathrm{Vol}(\alpha(\delta)K \cap \xi^{\perp})) < e^{-\frac{\delta^2}{2}} \mathrm{Vol}(\alpha(\delta)L \cap \xi^{\perp})), \;\; \forall \xi \in S^{n-1},$$

and so

$$\gamma_{n-1}(\alpha(\delta)K \cap \xi^{\perp}) < \gamma_{n-1}(\alpha(\delta)L \cap \xi^{\perp}), \;\; \forall \xi \in S^{n-1}.$$

Finally, choosing $\delta < \min\{\delta_0, \delta_1\}$ we get a counterexample to GBP in $\mathbb{R}^n$, $n \geq 5$.

Remark 2: The method described in Remark 1, can be used to solve the following dual version of GBP, which is a Gaussian analog of the Shephard problem (see [21] or [6], p. 140):

Assume $K,L \in \mathcal{C}_n$ *and*

$$\gamma_{n-1}(K|\xi^{\perp}) \leq \gamma_{n-1}(L|\xi^{\perp}), \;\; \forall \xi \in S^{n-1},$$

where $K|\xi^{\perp}$ *denotes the projection of* K *to the hyperplane orthogonal to* ξ*. Does it follow that*

$$\gamma_n(K) \leq \gamma_n(L)?$$

The answer to this problem is affirmative if $n = 2$. This can easily be shown using monotonic properties of the Gaussian measure. On the other hand, the answer is negative if $n \geq 3$. To show this one needs to apply the argument of Remark 1, using a counterexample to the original Shephard problem (see [20], [19] or [15]).

Remark 3: One can use a similar method to provide counterexamples to the Busemann-Petty problem in $\mathbb{R}^n$, $n \geq 5$ for a more general class of measures:

$$\gamma_{n,p}(K) = C_{n,p} \int\limits_K e^{-\|x\|_p^p/p} dx, \;\; p \geq 1,$$

where $\|x\|_p$ denotes the standard ℓ_p norm on $\mathbb{R}^n$. We also note that $\gamma_{n,p}$ ($p \neq 2$) is not rotation invariant, so the proofs of Theorems 1, 3, 4 can not be immediately generalized, and the Busemann-Petty problem for $\gamma_{n,p}$, $p \neq 2$ is open in $\mathbb{R}^3$ and $\mathbb{R}^4$.

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