Introduction to Analysis 2
Home Work 5, due Friday, March 9.
Instructor: Prof. Artem Zvavitch
30 POINTS

Problem 1. *Consider functions* $f_n(x) = e^{-nx^2}$. *Find function* $f(x)$ *such that* $f_n(x)$ *converges to* $f(x)$. *Show that* $f_n(x)$ *does not converges uniformly to* $f(x)$.

Problem 2. *Assume that* $f_n(x)$ *converges uniformly to* $f(x)$ *on* $[a, b]$, *prove that*

$$\lim_{n\to\infty} \int_a^b |f(x) - f_n(x)| = 0.$$

Show that converse is not true.

Problem 3. *Consider a sequence of functions* $f_n(x)$, *such that each* $f_n(x)$ *is a bounded function: if* $f_n(x)$ *converges pointwise to* $f(x)$ *is it true that* $f(x)$ *is a bounded function? if* $f_n(x)$ *converges uniformly to* $f(x)$ *is it true that* $f(x)$ *is a bounded function?*

Problem 4. *Find a sequence of continuous functions convergent pointwise to function* $f(x) = sign(x)$ *(*$f(x) = 1$, *is* $x > 0$, $f(x) = -1$, *is* $x < 0$ *and* $f(0) = 0$*).*

Problem 5. *Consider* $g(x) : R \to R$, *twice differentiable such that* $g''(x) = g(x)$ *and* $g(0) = g'(0) = 0$. *Prove that* $g(x) = 0$.

Problem 6. *Consider equation* $f''(x) = f(x)$ *for all* $x \in R$*:*

- *Show that* e^x *and* e^{-x} *are solutions and thus hyperbolic cosine* $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ *and hyperbolic sine* $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ *are also solutions.*
- *Prove that ANY solution of the above equation can be written as* $f(x) = \alpha \cosh(x) + \beta \sinh(x)$, *where* $\alpha, \beta \in R$.
- *Show that* $\cosh(x)$ *is even and* $\sinh(x)$ *is odd.*
- *Prove that*

$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$
$$\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y).$$