

Analysis 1 (42001/52001)
Home Work 7 , due on Friday October 31.
Instructor: Prof. Artem Zvavitch.

Problem 1. Show that $\sum_{i=1}^n \frac{(-1)^n}{\sqrt{n}}$ is convergent.

Problem 2. Show that $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ is divergent.

Hint: Use the method we used to study $\sum \frac{1}{n^p}$: split natural numbers into powers of two. More precisely: consider

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} = \frac{1}{2 \log 2} + \frac{1}{3 \log 3} + \frac{1}{4 \log 4} + \frac{1}{5 \log 5} + \dots$$

and notice that

$$\frac{1}{3 \log 3} + \frac{1}{4 \log 4} > \frac{1}{4 \log 4} + \frac{1}{4 \log 4} = \frac{2}{2^2 \log 2^2} = \frac{1}{2 \log 2^2}$$
$$\frac{1}{5 \log 5} + \frac{1}{6 \log 6} + \frac{1}{7 \log 7} + \frac{1}{8 \log 8} > \frac{4}{8 \log 8} = \frac{1}{2 \log 2^3}$$

And in general

$$\frac{1}{(2^{k-1} + 1) \log(2^{k-1} + 1)} + \dots + \frac{1}{2^k \log 2^k} > \frac{2^{k-1}}{2^k \log 2^k} = \frac{1}{2 \log 2^k} = \frac{1}{k} * \frac{1}{2 \log 2}$$

So finally

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} > \sum_{k=1}^{\infty} \frac{1}{k} * \frac{1}{2 \log 2},$$

and I think you can finish from here.

Problem 3. Assume $\sum_{n=1}^{\infty} a_n$ is convergent and $a_n > 0$ is it true that

$\sum_{n=1}^{\infty} \sqrt{a_n}$ is also convergent? What can you say about $\sum_{n=1}^{\infty} a_n^2$?

Problem 4. Assume $\sum_{n=1}^{\infty} a_n$ is convergent is it true that $\sum_{n=1}^{\infty} |a_n|$ is convergent?

Problem 5. Use $\varepsilon - \delta$ -definition of limit to show that

- $\lim_{x \rightarrow 1} (x^2 + 1) = 2$
- $\lim_{x \rightarrow 1} \frac{x+1}{2-x} = 2$
- $\lim_{x \rightarrow 1} \frac{x^2-1}{1-x} = -2$