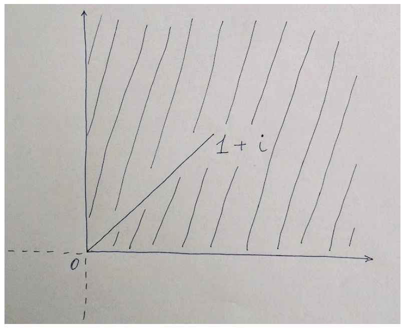
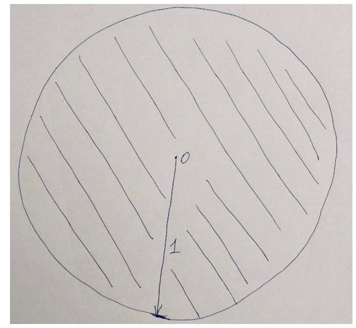
Functions of Complex Variables 1
Home Work 11, due on Wednesday April 11.

Problem 1.

Map 

conformally to 

Problem 2. *Prove that* $f(z) = -\frac{1}{2}(z + \frac{1}{z})$ *is a conformal map from the half-disc* $\{z = x + iy : |z| < 1, y > 0\}$ *to the upper half-plan.* **Hint:** *study quadratic equation* $f(z) = \omega$ *for* $\omega \neq 1, -1$.

Problem 3. *Consider a function* $f : \mathbb{D} \to \mathbb{D}$, *we say that* ω *is a fixed point of* f *if* $f(\omega) = \omega$.

- *Prove that if* f *is analytic and has two distinct fixed points, then* f *is the identity, i.e.* $f(z) = z$ *for all* $z \in \mathbb{D}$.
- *Is it true that every holomorphic function* $f : \mathbb{D} \to \mathbb{D}$ *have a fixed point?*

Problem 4. *A bit more story around Schwartz lemma:*

(1) *Show that consider a holomorphic function* $f : \mathbb{D} \to \mathbb{D}$. *Prove that*

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2},$$

This result is called the Schwartz-Pick lemma.

(2) *Consider a holomorphic function* $f(z)$ *such that* f *maps upper half-plane into upper half-plane (i.e.* $Imf(z) > 0$ *for all* $Imz > 0$*). Prove that*

$$\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|}$$

and

$$\frac{|f'(z)|}{Imf(z)} \leq \frac{1}{Imz}.$$

(3) *In 1) and 2) prove that equality implies that* $f(z)$ *is a fractional linear transformation.*

(4) *Derive the corresponding inequalities if* $f(z)$ *maps* $\mathbb{D}$ *into the upper half plane.*

(5) *Prove by use of Schwarz's lemma that every one-to-one conformal mapping of a disk onto another (or a half plane) is given by a fractional linear transformation.*