Functions of Complex Variables 1
Home Work 12, due on Wednesday April 18.

Problem 1. *Reminder: we say that $S(z)$ is a linear fractional transformation if $S(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.*

- *Prove that the reflection $z \to \bar{z}$ is a not a fractional linear transformation.*
- *Show that any fractional transformation which transforms the real axis into itself can be written with real coefficients.*

Problem 2. *Map the open region between $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ on a half plane.*

Problem 3. *Let Ω, z_0 and f be given by the Riemann mapping theorem.*

- *Assume $z_0 \in \mathbb{R}$ and Ω is symmetric with respect to the real line. Show that $f(\bar{z}) = \overline{f(z)}$.*
- *What is the corresponding conclusion if Ω is symmetric with respect to the point z_0?*

Problem 4. *The angle between two non-zero complex numbers z and ω (taken in this order) is the oriented angle in $(-\pi, \pi]$ that is formed between the two vectors in $\mathbb{R}^2$ corresponding to points z and ω. Assume α is such an angle. Please, find formulas, using the scalar product between vectors in $\mathbb{R}^2$, for $\cos\alpha$ and $\sin\alpha$.*

Problem 5. *Consider two smooth curves $\gamma : [a, b] \to \mathbb{C}$ and $\nu : [a, b] \to \mathbb{C}$. Assume, those curves intersect at point $z_0 = \gamma(t_0) = \nu(t_0)$, we say that the curves intersect at z_0 at angle α if α is the angle between their tangent lines at z_0, i.e. the angle between vectors $\gamma'(t_0)$ and $\nu'(t_0)$ (assuming, both are non-zero!).*

We say that the holomorphic function f preserve angles at z_0 if for **any** *two curves γ, ν as above the angle between curves $f(\gamma)$ and $f(\nu)$ at $f(z_0)$ is equal to the angle between γ and ν at z_0 (in particular, again, we do assume that the tangent vectors to $\gamma, \nu, f(\gamma), f(\nu)$ at time t_0 are all non-zero:*

- *Prove that if $f : \Omega \to \mathbb{C}$ is holomorphic, and $f'(z_0) \neq 0$, then f preserves angles at z_0.*
- *Conversely, prove that if $f : \Omega \to \mathbb{C}$ is a complex-valued function that is real-differentiable at $z_0 \in \Omega$ and $J_f(z_0) \neq 0$ (with non-zero Jacobian at z_0), such that f preserves angles at z_0, then f is holomorphic at z_0 with $f'(z) \neq 0$.*