

**Functions of Complex Variables 1**  
**Home Work 1, due on Wednesday January 24.**

**Problem 1.** Please, provide geometric description for set of points  $z \in \mathbb{C}$  defined by

- $r < |z - z_0| < R$ , for some  $R > r > 0$  and  $z_0 \in \mathbb{C}$ .
- $|z - z_1| = |z - z_2|$ , for some  $z_1, z_2 \in \mathbb{C}$ .
- $|z| = \operatorname{Re}(z) + 2$

**Problem 2.** Please, simplify  $\sum_{k=1}^n \sin kx$ .

**Problem 3.** Let  $w = se^{i\theta}$ , where  $s > 0$  and  $\theta \in \mathbb{R}$ . Please, solve the equation  $z^n = w$  in  $\mathbb{C}$ , where  $n$  is a fixed natural number.

**Problem 4.** Compute

- $\sqrt{i}, \sqrt{-i}, \sqrt{1+i}, \sqrt{\frac{1-i\sqrt{3}}{2}}$ .
- $\sqrt[4]{i}, \sqrt[4]{-i}$ .

**Problem 5.** In this problem we will introduce a family of mappings called **Blaschke factors**.

- Let  $z, w$  be two complex numbers such that  $\bar{z}w \neq 1$ . Prove that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| < 1 \text{ if } |z| < 1 \text{ and } |w| < 1.$$

Also show that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| = 1 \text{ if } |z| = 1 \text{ and } |w| = 1.$$

**Hint:** Show that we can assume that  $z$  is real. Then show that it is sufficient to prove that

$$(r - w)(r - \bar{w}) \leq (1 - rw)(1 - r\bar{w})$$

for appropriate  $r$  and  $|w|$ .

- Prove that for a fixed  $w$  in the unit disc  $\mathbb{D}$ , the mapping

$$F(z) = \frac{w - z}{1 - \bar{w}z}$$

satisfies the following conditions:

- (1)  $F$  maps the unit disk to itself and is holomorphic.
- (2)  $F$  interchanges 0 and  $w$ , namely  $F(0) = w$  and  $F(w) = 0$ .
- (3)  $|F(z)| = 1$  for  $|z| = 1$ .
- (4)  $F : \mathbb{D} \rightarrow \mathbb{D}$  is bijection. (it may help to compute  $F \circ F$ .)

**Problem 6.** Prove a complex version of the chain rule (**you may prove just one of two formulas below**): Suppose  $U$  and  $V$  are open sets in the complex plane. Prove that if  $f : U \rightarrow V$  and  $g : V \rightarrow \mathbb{C}$  are two functions that are differentiable (in the real sense, i.e., as functions of the two real variables  $x$  and  $y$ ) and let  $h = g \circ f$ , then

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z} \frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}} \frac{\partial f}{\partial z}$$

and

$$\frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial z} \frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}} \frac{\partial f}{\partial \bar{z}}.$$