Functions of Complex Variables 1
Home Work 2, due on Wednesday January 31.

Problem 1. *Consider a non-constant analytic function* $f : \mathbb{C} \to C$ *show that such a function can not have a constant absolute value.*

Problem 2. *Show that in polar coordinates, the Cauchy-Riemann equations take the form*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta, \text{ where } z = re^{i\theta}, \text{ with } \theta \in (-\pi, \pi)$$

is holomorphic in the region $r > 0$ *and* $\theta \in (-\pi, \pi)$.

Problem 3. *Show that*

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \triangle,$$

where $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ *is the Laplacian.*

Use the above formula to prove that if f *is holomorphic in a open set* Ω, *then the real and imaginary parts of* f *are* **harmonic**; *that is their Laplacian is zero.*

Problem 4. *Proof that* f *is analytic if and only if* $\overline{f(\bar{z})}$ *is analytic.*

Problem 5. *Assume* $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = R$, *prove that* R *is the radius of convergence of* $\sum a_n z^n$.

Problem 6. *Find the radius of convergence of* $\sum nz^n$, $\sum z^n/n^2$ *and* $\sum z^n/n$ *now show that*

- $\sum nz^n$ *does not converge on any point of the unit circle.*
- $\sum z^n/n^2$ *converges at every point of the unit circle.*
- $\sum z^n/n$ *converges at every of the point of unit circle but* $z = 1$.

Note that you may use Summation by Parts trick to study $\sum z^n/n$. *The trick is to note that if* $\{a_k\}$ *and* $\{b_k\}$ *are two sequences then*

$$\sum_{k=m}^{n} a_k(b_{k+1} - b_k) = (a_n b_{n+1} - a_m b_m) - \sum_{k=m+1}^{n} b_k(a_k - a_{k-1}).$$