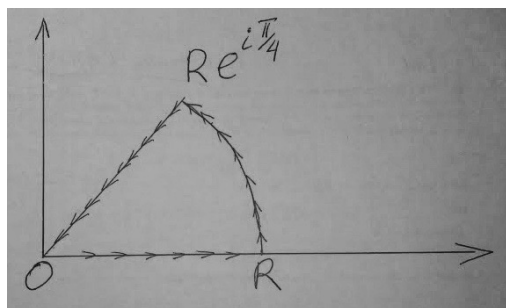


Functions of Complex Variables 1
Home Work 4, due on Wednesday February 14.

Problem 1. Prove that

$$\int_0^{\infty} \sin(x^2) dx = \int_0^{\infty} \sin(x^2) dx = \frac{\sqrt{2\pi}}{4}.$$

Hint: integrate function e^{-z^2} over the path



Problem 2. Show that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$. **Hint:** The integral equals $\frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{ix}-1}{x} dx$.

Problem 3. Compute integrals

- $\int_{|z|=1} \frac{e^z}{z} dz,$
- $\int_{|z|=2} \frac{dz}{z^2+1},$
- $\int_{|z|=2} (z-1)^{-4} e^z dz.$

Problem 4. Consider an entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ such that there exists $n \in \mathbb{N}$ and $R > 0$ with the property $|f(z)| < |z|^n$ for all $|z| > R$. Show that then f is a polynomial.

Problem 5. Does there exist an analytic function f such that, there exists point $z \in \mathbb{C}$, f is analytic at z and $|f^{(n)}(z)| > n!n^n$?

Problem 6. Suppose f is an entire function such that for every $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{i=0}^{\infty} c_n(z-z_0)^n$$

is equal to 0. Prove that f is a polynomial.

Problem 7. Classical Weierstrass's theorem from real analysis tell us that a continuous function on $[0, 1]$ can be uniformly approximated by polynomials. Can every continuous function on the closed unit disc be approximated uniformly by polynomials in the variable z ?

Problem 8. (MOVED TO NEXT HW) Let D be a unit disc centered at the origin and \bar{D} be the closure of D . Assume that $f(z) \neq 0$ for all $z \in \bar{D}$ and $f(z)$ is continuous on \bar{D} and holomorphic in D . Prove that if $|f(z)| = 1$ for all $|z| = 1$

2

then f is a constant. **Hint:** Extend f to all \mathbb{C} by $f(z) = \frac{1}{f(\bar{z}^{-1})}$ for $|z| > 1$, and use Schwartz reflection principle.