Functions of Complex Variables 1
Home Work 5, due on Wednesday February 21.

Problem 1. *Let D be a unit disc centered at the origin. Suppose that $f : D \to \mathbb{C}$ is holomorphic. Show that*

$$2|f'(0)| \leq \sup_{z,w \in D} |f(z) - f(w)|.$$

Problem 2. *Show that every function f which is analytic in a symmetric region Ω can be written in the form $f_1 + if_2$, where f_1, f_2 are analytic in Ω and real on the real axis.*

Problem 3. *Let D be a unit disc centered at the origin and $\bar{D}$ be the closure of D. Assume that $f(z) \neq 0$ for all $z \in \bar{D}$ and $f(z)$ is continuous on $\bar{D}$ and holomorphic in D. Prove that if $|f(z)| = 1$ for all $|z| = 1$ then f is a constant.* **Hint:** *Extend f to all $\mathbb{C}$ by $f(z) = \frac{1}{\overline{f(\bar{z}^{-1})}}$ for $|z| > 1$, and* **use ideas** *of the proof from Schwartz reflection principle.*

Problem 4. "Stollen" from H. Boas: *Please, prove Runge theorem in a very specific case (you may use the ideas from the class):, Take Q to be the closed unit square in the complex plane, for example $Q = \{z \in \mathbb{C} : |Rez| \leq 1$ and $|Imz| \leq 1\}$. Assume that f is holomorphic in an open neighborhood of Q, then for any $\varepsilon > 0$ there exists a polynomial $p(z)$ such that $\max_{z \in Q}\{|f(z) - p(z)| < \varepsilon\}$. Please show that*

- *There is a slightly larger square Q' such that $Q \subset Q'$ and by Cauchy's formula, we can express $f(z)$ for $z \in Q$ as an integral over the boundary of the larger square.*
- *By approximating the integral by a Riemann sum, we can approximate $f(z)$ for $z \in Q$ by a finite linear combination of functions of the form*

$$\frac{1}{(a_n - z)},$$

  where the a_n are points outside Q. You need to verify that the approximation depends on z in a uniform way.

At this point, we have approximated f by a rational function with singularities outside Q (but close to Q). We need to push the singularities far away from Q.

- *The trick, as we done in class, for doing this is to write*

$$\frac{1}{z-a} = \frac{1}{z-b} \cdot \frac{1}{1 + \frac{b-a}{z-b}}$$

  and to truncate a convergent geometric series expansion.
- *When $a \notin Q$, the function $\frac{1}{(z-a)}$ can be approximated uniformly on Q by a linear combination of powers of $\frac{1}{(z-b)}$, where b is farther away from Q than a is.*
- *If b is far enough away from Q, then $\frac{1}{(z-b)}$ can be replaced by a partial sum of its Taylor series to finish the construction of a polynomial approximation of f on Q.*