

Functions of Complex Variables 1
Home Work 6, due on Wednesday February 28.

Problem 1. Using $\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$, show that the complex zeros of $\sin \pi z$ are exactly at the integers, and that they are each of order 1.

Problem 2. Show that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh \pi x} dx = \frac{1}{\cosh \pi \xi}.$$

Problem 3. Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \pi.$$

Problem 4. Prove that for $a, b \in \mathbb{R}$, such that $a > |b|$:

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

Problem 5. Suppose $f(z)$ is holomorphic in a punched disc $D_r(z_0) - \{z_0\}$. Suppose also that

$$|f(z)| \leq A|z - z_0|^{-1+\varepsilon},$$

for some $\varepsilon > 0$, $A > 0$ and all z near z_0 . Show that the singularity of f at z_0 is removable.

Problem 6. Prove that all entire functions that are also injective take the form $f(z) = az + b$, with $a, b \in \mathbb{C}$ and $a \neq 0$. **Hint:** You may apply the Casorati-Weierstrass theorem to $f(1/z)$.