Functions of Complex Variables 1
Home Work 7, due on Wednesday March 6.

Problem 1. *Use Rouche's theorem to compute how many roots does the equation*

$$z^7 - 2z^5 + 6z^3 - z + 1 = 0$$

have in the disc $|z| < 1$*?*

Problem 2. *Prove that if* f *is an entire function that satisfies*

$$\max_{|z|=R} |f(z)| \leq C_1 R^k + C_2$$

for all $R > 0$ *and some non-negative integer* k *and constants* $C_1, C_2 > 0$*, then* f *is a polynomial of degree at most* k*.*

Problem 3. *Show that if the real part of an entire function* f *is bounded, then* f *is a constant.*

Problem 4. *Let* f *be non-constant and holomorphic in an open set containing the closed unit disc* $\bar{D} = \{z \in \mathbb{C} : |z| \leq 1\}$*.*

- *Show that if* $|f(z)| = 1$*, for all* z *such that* $|z| = 1$*, then the image of* f *contains the unit disc* $D = \{z \in \mathbb{C} : |z| < 1\}$*.* **Hint:** *You need to show that* $f(z) = \omega_0$ *has a root for every* $\omega_0 \in D$*. You may use maximum modulus principle to do so, in addition with showing first that it is enough to prove that there is a solution for* $f(z) = 0$*.*
- *If* $|f(z)| \geq 1$ *for all* z *such that* $|z| = 1$*, and there exists a point* $z_0 \in D$ *such that* $|f(z_0)| < 1$*, then the image of* f *contains the unit disc.*

Problem 5. *Prove the maximum principle for harmonic functions (see HW 2): Let* u *be a real-valued function defined on the unit disc* D*. Suppose that* u *is twice differential and harmonic, i.e.*

$$\triangle u(x, y) = 0,$$

for all $(x, y) \in D$*, where* $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ *is the Laplacian.*

- *Prove that there exists a holomorphic function* f *on the unit disc such that*

  $$Re(f) = u.$$

  Also show that the imaginary part of f *is uniquely defined up to an additive (real) constant.* **Hint:** *Use that we have shown* $f'(z) = 2\partial u/\partial z$*. Therefore, define* $g(z) = 2\partial u/\partial z$ *and prove that* g *is holomorphic. Next show that one can find* F *such that* $F' = g$*. Finally show that* $Re(F)$ *differs from* u *by a real constant.*
- *If* u *is a non-constant real-valued harmonic function in a region* Ω *then* u *cannot attain a maximum (or minimum) in* Ω*.* **Hint:** *Assume that* u *attains a local maximum at* z_0*. Let* f *be holomorphic near* z_0 *with* $u = Re(f)$*, and show that* f *is not open.*
- *Suppose that* Ω *is a region with compact* $\bar{\Omega}$*. If* u *is harmonic in* Ω *and continuous in* $\bar{\Omega}$*, then*

  $$\sup_{z \in \Omega} |u(z)| \leq \sup_{\bar{\Omega} - \Omega} |u(z)|.$$