

**Functions of Complex Variables 1**  
**Home Work 8, due on Wednesday March 13.**

**Problem 1.** Please, give a definition and study the provide properties of  $\arcsin z$  and  $\arccos z$ , the inverse functions of  $\sin z$  and  $\cos z$ .

**Problem 2.** Please,

- Find real and imaginary parts of  $\exp(e^z)$  and  $z^z$ .
- Determine all values of  $2^i$ ,  $(-1)^{2i}$  and  $i^{-\pi i}$ .

**Problem 3.** Please show that the mean square convergence dominates the uniform convergence of analytic functions (**Hint:** you may use the mean-value property). More precisely, consider an open set  $U$  is an open subset of  $\mathbb{C}$  define

$$\|f\|_{L_2(U)} = \left( \int_U |f(z)|^2 dx dy \right)^{1/2}$$

for the mean square norm (yes, the integral is "real"), and

$$\|f\|_{L^\infty(U)} = \sup_{z \in U} |f(z)|$$

for the sup norm.

- If  $f$  is holomorphic in a neighborhood of the disc  $D_r(z_0)$ , show that for any  $0 < s < r$  there exists a constant  $C > 0$  (which may depend on  $s$  and  $r$ ) such that

$$\|f\|_{L^\infty(D_s(z_0))} \leq C \|f\|_{L_2(D_r(z_0))}.$$

- Prove that if  $\{f_n\}_{n=1}^\infty$  is a Cauchy sequence of holomorphic functions in the mean square norm  $\|\cdot\|_{L_2(U)}$  then the sequence  $\{f_n\}_{n=1}^\infty$  converges uniformly on every compact subset of  $U$  to a holomorphic function.

**Problem 4.** .

- Prove that every open convex open set is simply connected.
- Prove that every star-shaped open set is simply connected (we say that  $\Omega$  is star-shaped if there exists  $z_0 \in \Omega$  so that for any  $[z_0, z] \subset \Omega$ , for any other  $z \in \Omega$ , where  $[z_0, z]$  is an interval connecting  $z_0$  and  $z$ ).
- Is it true that every simply connected set is star-shaped?
- Is it true that the union of simply connected sets is simply connected?

**Problem 5.** Show that there is **no** holomorphic function  $f$  in the unit disc  $D = \{z \in \mathbb{C} : |z| < 1\}$  that extends continuously to  $\partial D$  such that  $f(z) = 1/z$  for  $z \in \partial D$ .