Introduction to Analysis
VERY BIG Home Work + HW 10, due THE DAY OF FINAL EXAM.
Instructor: Prof. Artem Zvavitch
There are 22 problems 5 points each!

Problem 1. *Use the principal of mathematical induction to show that for any numbers b and $q \neq 1$ if*

$$x_i = b \cdot q^i,$$

then

$$\sum_{i=0}^{n} x_i = b\frac{q^{n+1}-1}{q-1}.$$

Please, also find

$$\lim_{n\to\infty} \sum_{i=0}^{n} x_i.$$

Problem 2. *Consider the set*

$$S = \{[x,y]\times[c,d], \textit{where } x,y,c,d \textit{ are rational numbers}\}.$$

Show that S is a countable set.

Problem 3. *Prove that*

$$\sqrt{|a|+|b|} \leq \sqrt{|a|} + \sqrt{|b|}.$$

Can you generalize this fact? I.e. find all p for which

$$(|a|+|b|)^p \leq |a|^p + |b|^p.$$

Problem 4. *Show that $|x-5|$ is a continuous function on R. Is it uniformly continuous?*

Problem 5. *Show that $(x-5)^2$ is a continuous function on R. Is it uniformly continuous?*

Problem 6. *Prove that*

$$\sup_{x\in A}|f(x)+g(x)| \leq \sup_{x\in A}|f(x)| + \sup_{x\in A}|g(x)|,$$

and

$$\sup_{x\in A}|f(x)\times g(x)| \leq \sup_{x\in A}|f(x)| \times \sup_{x\in A}|g(x)|.$$

Problem 7. *Prove that there exists a positive real number x so that $x^2=5$. Please, also prove that $2<x<2.5$.*

Please, find a sequence $\{x_n\}_{n=1}^{\infty}$ of rational numbers such that

$$\lim_{n\to 1} x_n = \sqrt{5}.$$

Problem 8. *It it true that the sum of two rational numbers is rational?*

Is it true that the sum of irrational number and rational number is necessary irrational?

Is it true that the sum of two irrational numbers is necessary irrational?

Do not forget to show a proof, if the answer to any of questions is Yes

Problem 9. *Use the definition of limit to show that*

$$\lim_{n\to\infty} \frac{21+n}{-8-n} = -1$$

and

$$\lim_{n\to\infty} \frac{n^2+1}{n} = \infty.$$

Problem 10. *Show that*

$$a_n = \sum_{i=0}^{n} \frac{(-1)^i}{3^i}$$

is a convergent sequence.

Problem 11. *Show that if $a_{n+1} = a_n + n^{-n}$, then a_n is a convergent sequence.*

Problem 12. *Let $x_1 = 0$ and $x_{n+1} = \frac{1}{4}(x_n + 1)$ show that a) x_n is bounded. b) Monotone c) convergent. Finally find the limit of x_n.*

Problem 13. *Prove that if a_n is a Cauchy sequence then $|a_n|$ is also a Cauchy sequence. Is it true that if $|b_n|$ is a Cauchy sequence then b_n is a Cauchy sequence? Is it true that b_n has a convergent subsequence?*

Problem 14. *Use $\varepsilon - \delta$ definition of limit to show that*

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4.$$

Problem 15. *Let $f(x)$ be a continuous function such that $f(2^{-n}) = \frac{n+1}{n}$, find $f(0)$.*

Problem 16. *Show that*

$$\lim_{x\to 3} \frac{1}{x-3}$$

does not exists.

Problem 17. *Prove that if $f(x)$ and $g(x)$ are uniformly continuous functions on A, then $f(x) + g(x)$ is also uniformly continuous on A.*

Problem 18. *Let $f(x)$ be a continuous function on $[a, b]$, such that $f(a) \times f(b) \leq 0$, show that then there exist $c \in [a, b]$ such that $f(c) = 0$.*

Problem 19. *Let $f(x)$ be a continuous function on $[0, 1]$ such that $f(0) = -1$ and $f(1) = 2$, show that there exist $x_0 \in [0, 1]$, such that $f(x_0) = x_0$.*

Problem 20. *Show that $q(x) = -3x^2$ is a Lipschitz function on $[0, 1]$. Show that if f is a Lipschitz function with Lipschitz constant L, then $g(x) = -3f(x)$ is a Lipschitz function with constant 3L.*

Problem 21. *Assume $f(x)$ and $g(x)$ are monotone increasing functions on $[0, 1]$, is it true that $f(x) \times g(x)$ is also monotone increasing?*

Problem 22. *Assume that $f(x)$ is a strictly increasing function on $[a, b]$, show that $f^{-1}(x)$ (inverse of f) is also an increasing function on $[f(a), f(b)]$.*