Introduction to Analysis 2 (42001/52001)
HW7, due April 13
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Problem 1. *Let*

$$f(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}.$$

Prove that

$$\int_0^{\pi/2} f(x)dx = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^3}.$$

Problem 2. *Find the largest set on which* $\sum_{k=1}^{\infty} \frac{x^{3k}}{k}$ *is convergent.*

Problem 3. *Put* $P_0 = 0$ *and define, for* $n = 0, 1, 2, \dots$

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Prove that $P_n(x)$ *converge uniformly to* $|x|$ *on* $[-1, 1]$.

HINT: First show that

$$|x| - P_{n+1}(x) = (|x| - P_n(x)) \left(1 - \frac{|x| + P_n(x)}{2}\right).$$

Next show that $0 \le P_n(x) \le P_{n+1}(x) \le |x|$ for $x \in [-1, 1]$, and that

$$|x| - P_n(x) \le |x| \left(1 - \frac{|x|}{2}\right)^n \le \frac{2}{n+1}$$

for $x \in [-1, 1]$.

Problem 4. *Consider a collection of sets* $\mathcal{R}$ *such that for any* $A, B \in \mathcal{R}$ *we get*

$$A \cap B \in \mathcal{R} \text{ and } A \cup B \in \mathcal{R}.$$

Show that $\mathcal{R}$ *is not necessary a ring.*

Problem 5. *Prove that the set all of finite segments (open, closed, semi-open) (we also include empty set) is not a ring. Also prove that a collection of all finite unions of those segments is a ring.*