

Introduction to Analysis II (42001/52001)
HW7, due Thursday, March 12
Instructor: Prof. Artem Zvavitch

Problem 1 Consider $x > 0$. Prove that

$$\int_0^x t^2 e^{-\frac{t^2}{2}} dt \leq x(1 - e^{-\frac{x^2}{2}}).$$

Problem 2 Show that the function

$$f(x) = \int_0^x e^{t^2} dt$$

is a Lipschitz function.

Problem 3 Find all continuous functions f such that

$$\int_0^x f(t) dt = \int_0^{2x} f(t) dt, \text{ for all } x \in \mathbb{R}.$$

Problem 4 Consider functions $f, g \in \mathcal{R}[a, b]$

- Show that $\int_a^b (tf \pm \frac{1}{t}g)^2 dx \geq 0$ for all $t \in \mathbb{R}$.
- Use previous fact to show that

$$2 \left| \int_a^b (fg) \right| \leq t \int_a^b f^2 + \frac{1}{t} \int_a^b g^2.$$

- If $\int f^2 = 0$, show that $\int_a^b fg = 0$.
- Now, please, prove one of the most famous inequalities in Analysis
Cauch-Bunyakowski-Schwartz Inequality:

$$\left(\int_a^b |fg| \right)^2 \leq \left(\int_a^b f^2 \right) \left(\int_a^b g^2 \right).$$

- Finally, prove the triangle inequality for functions

$$\sqrt{\int_a^b (f - g)^2} \leq \sqrt{\int_a^b f^2} + \sqrt{\int_a^b g^2}.$$