Introduction to Analysis 1
HW7 and HW 8, due Thursday, November 13
40 points.
Instructor: Prof. Artem Zvavitch

Problem 1. *Show that* $\sum_{i=1}^{n} \frac{(-1)^n}{\sqrt{n}}$ *is convergent.*

Problem 2. *Use the Cauchy Test to show that* $\sum \frac{1}{n \log n}$ *is divergent.*

Hint: The idea of Cauchy Test is something that we used a number of times before when where proving divergence/convergence of $\sum \frac{1}{n^p}$: split natural numbers into powers of two. More precisely: consider

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} = \frac{1}{2 \log 2} + \frac{1}{3 \log 3} + \frac{1}{4 \log 4} + \frac{1}{5 \log 5} + ...$$

and notice that

$$\frac{1}{3 \log 3} + \frac{1}{4 \log 4} > \frac{1}{4 \log 4} + \frac{1}{4 \log 4} = \frac{2}{2^2 \log 2^2} = \frac{1}{2 \log 2^2}$$

$$\frac{1}{5 \log 5} + \frac{1}{6 \log 6} + \frac{1}{7 \log 7} + \frac{1}{8 \log 8} > \frac{4}{8 \log 8} = \frac{1}{2 \log 2^3}$$

And in general

$$\frac{1}{(2^{k-1}+1) \log(2^{k-1}+1)} + \cdots + \frac{1}{2^k \log 2^k} > \frac{2^{k-1}}{2^k \log 2^k} = \frac{1}{2 \log 2^k} = \frac{1}{k} * \frac{1}{2 \log 2}$$

So finally

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} > \sum_{k=1}^{\infty} \frac{1}{k} * \frac{1}{2 \log 2},$$

and I think you can finish from here.

Problem 3. *Show that the series* $\sum_{n=1}^{\infty} \frac{cos n}{2^n}$ *is convergent.*

Problem 4. *Assume* $\sum_{n=1}^{\infty} a_n$ *is convergent and* $a_n > 0$ *is it true that* $\sum_{n=1}^{\infty} \sqrt{a_n}$ *is also convergent? What can you say about* $\sum_{n=1}^{\infty} a_n^2$*?*

Problem 5. *Assume* $\sum_{n=1}^{\infty} a_n$ *is convergent is it true that* $\sum_{n=1}^{\infty} |a_n|$ *is convergent?*

Problem 6. *Use* $\varepsilon - \delta$*-definition of limit to show that*

- $\lim_{x \to 1} (x^2 + 1) = 2$
- $\lim_{x \to 1} \frac{x+1}{2-x} = 2$
- $\lim_{x \to 1} \frac{x^2-1}{1-x} = -2$

Problem 7. *Use sequential criterion to show that*

$$\lim_{x \to 0} \frac{1}{x^2}$$

does not exist.

Problem 8. *Use sequential criterion to show that*

$$\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$$

does not exist.

Problem 9. *Use $\varepsilon - \delta$-definition of limit to show that*

$$\lim_{x\to 0} x \sin\frac{1}{x} = 0.$$

Problem 10. *Use $\varepsilon - \delta$-definition of limit to show that*

$$\lim_{x\to c} x^4 = c^4.$$

Problem 11. *Use $\varepsilon - \delta$-definition of limit to show that if*

$$\lim_{x\to c} f(x) = L$$

then

$$\lim_{x\to c} f(x)^2 = L^2.$$

Show that converse statement is no longer true.

Problem 12. (Extra 10pts) *Prove that*

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$