Introduction to Analysis 2 (42001/52001)
HW8 AND HW9, due April 23
Instructor: Prof. Artem Zvavitch

Problem 1. *Let $A \subset X$. A function*

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A. \end{cases}$$

defined for $x \in X$, is called a **characteristic function of a set** *A. Prove, that for any two sets $A, B \subset X$:*

- $1_{A\cap B}(x) = 1_A(x)1_B(x)$.
- $1_{A\cup B}(x) = 1_A(x) + 1_B(x) - 1_{A\cap B}(x)$.
- $1_{A\setminus B}(x) = 1_A(x) - 1_A(x)1_B(x)$.
- $1_{A\delta B}(x) = 1_A(x) + 1_B(x) - 21_A(x)1_B(x)$.

Definition Let $E := [0,1] \times [0,1]$, and let $A \subset E$. The **inner measure** $\mu_*(A)$ (of A) is a number defined as

$$\mu_*(A) := 1 - \mu^*(E \setminus A).$$

Problem 2. *Prove that $\mu_*(A) \le \mu^*(A)$.*
Hint: *Show/Use* $\mu^*(E \setminus A) + \mu^*(A) \ge \mu^*(E)$.

Problem 3. *Prove that $A \subseteq E$ is Lebesgue measurable if and only if $\mu_*(A) = \mu^*(A)$.*
Hint: *Let $A \subseteq E$ be Lebesgue measurable. Then $\forall \epsilon > 0$ there exists an elementary set B such that $\mu^*(A \triangle B) < \epsilon$. Prove that*

$$\mu^*(B) - \epsilon \le \mu_*(A) \le \mu^*(A) \le \mu^*(B) + \epsilon.$$

Conversely, assume that $\mu_*(A) = \mu^*(A)$. Then prove the following chain of statements leading to the result.

a) $\forall \epsilon > 0$ there exist elementary sets P_n, Q_n, $n = 1, 2, \ldots$, such that

$$A \subset \cup_{n=1}^{\infty} P_n, \qquad \mu^*(A) \ge \sum_{n=1}^{\infty} m'(P_n) - \epsilon,$$

and

$$(E \setminus A) \subset \cup_{n=1}^{\infty} Q_n, \qquad \mu^*(E \setminus A) \ge \sum_{n=1}^{\infty} m'(Q_n) - \epsilon.$$

Conclude that

$$\sum_{n=1}^{\infty} (m'(P_n) + m'(Q_n)) \le 1 + 2\epsilon,$$

and that there exists N such that

$$\sum_{n=N+1}^{\infty} (m'(P_n) + m'(Q_n)) < \epsilon.$$

b) Denote

$$\cup_{n=1}^{\infty} P_n = P, \qquad \cup_{n=1}^{\infty} Q_n = Q, \qquad \cup_{n=1}^{N} P_n = P_N \qquad \cup_{n=1}^{N} Q_n = Q_N.$$

Observe that

$$\mu^*(P_N \triangle A) \leq \mu^*(P_N \setminus A) + \mu^*(A \setminus P_N).$$

c) Prove that

$$\mu^*(A \setminus P_N) \leq \sum_{n=N+1}^{\infty} m'(P_n).$$

d) Observe that

$$P_N \setminus A \subset (P_N \cap Q_N) \cup (P_N \cap (Q \setminus Q_n)) \subset (P_N \cap Q_N) \cup (Q \setminus Q_n),$$

and conclude

$$\mu^*(P_N \setminus A) \leq \mu^*(P_N \cap Q_N) + \sum_{n=N+1}^{\infty} m'(Q_n).$$

e) Observe that $E \subseteq (P \cup Q)$*, and show that*

$$1 = \mu^*(E) \leq \mu^*(P_N \cup Q_N) + \mu^*(P \setminus P_N) + \mu^*(Q \setminus Q_N).$$

Use the fact that for elementary sets C, D*,*

$$m'(C \cup D) = m'(C) + m'(D) - m'(C \cap D),$$

and a) to conclude that

$$1 \leq \sum_{n=1}^{\infty}(m'(P_n) + m'(Q_n)) - \mu^*(P_N \cap Q_N) \leq 1 + 3\epsilon - \mu^*(P_N \cap Q_N),$$

and $\mu^*(P_N \cap Q_N) \leq 3\epsilon$*.*

f) Use d) to show that

$$\mu^*(P_N \setminus A) \leq 2\epsilon + \sum_{n=N+1}^{\infty} m'(Q_n).$$

g) "Glue" pieces b), c), f) to obtain the desired result.