Introduction to Analysis II (42001/52001)
H8, due Thursday, March 19
EACH PROBLEM IS 5 points so you may get some extra points towards your exam Instructor: Prof. Artem Zvavitch

Problem 1. *Show that the sequence* $f_n(x) = nxe^{-nx^2/2}$ *converges uniformly on* (a, ∞)*, for any* $a > 0$*, but it does not converge uniformly on* $[0, \infty)$*.*

Problem 2. *Prove that if sequences* f_n*,* g_n *converge uniformly on* A *to* f *and* g*, resectively, then* $f_n + g_n$ *converges uniformly on* A *to* $f + g$*.*

Problem 3. *Show that the product of uniformly convergent sequences is not necessary uniformly convergent.* **Hint:** *consider* $f_n = x + \frac{1}{n}$ *and* f_n^2 *on* $\mathbb{R}$*.*

Problem 4. *Now let's repair the product property: Prove that if sequences of* **bounded functions** f_n*,* g_n *converge uniformly on* A *to* f *and* g*, resectively, then* $f_n g_n$ *converges uniformly on* A *to* fg*.*

Problem 5. *Show that the sequence* $x^n/(1 + x^n)$ *dooes not converge uniformly on* $[0, 2]$ *by showing that the limit function is not continuous on* $[0, 2]$*.*

Problem 6. *Let* $f_n(x) = e^{-nx}/n$ *for* $x \geq 0$*. Study the relation between* $\lim f_n$ *and* $\lim f_n'$*.*

Problem 7. *Assume that* $f_n(x)$ *converges uniformly to* $f(x)$ *on* $[a, b]$*, prove that then*

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)| = 0.$$

Show that converse is not true.

Problem 8. *Consider a sequence of bounded functions* $f_n(x)$*. If* $f_n(x)$ *converges pointwise to* $f(x)$ *is it true that* $f(x)$ *is a bounded function? if If* $f_n(x)$ *converges uniformly to* $f(x)$ *is it true that* $f(x)$ *is a bounded function?*

Problem 9. *Find a sequence of continuous functions convergent pointwise to function* $f(x) = \mathrm{sign}(x)$*.*