Introduction to Analysis II (42001/52001)
H8, due Thursday, April 9
YES YOU DO NOT NEED THIS HW BEFORE EXAM.
Instructor: Prof. Artem Zvavitch

Problem 1. *Consider twice differentiable function* $g(x) : \mathbb{R} \to \mathbb{R}$ *such that*

$$g''(x) = g(x) \text{ for all } x \in \mathbb{R}$$

and $g(0) = g'(0) = 0$. *Prove that* $g(x) = 0$.

Problem 2. *Consider equation* $f''(x) = f(x)$ *for all* $x \in \mathbb{R}$*:*

- *Show that* e^x *and* e^{-x} *are solutions for this equation and thus hyperbolic cosine* $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ *and hyperbolic sine* $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ *are solutions.*
- *Prove that any solution of the above equation can be written as*

$$f(x) = \alpha \cosh(x) + \beta \sinh(x), \text{ for some } \alpha, \beta \in \mathbb{R}.$$

- *Prove that* $\cosh(x)$ *is even and* $\sinh(x)$ *is odd functions.*
- *Probe that*

$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y).$$
$$\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y).$$

Problem 3. *Decide (with explanation!) which of the following series convergent and which divergent:*

- $\sum_{k=1}^{\infty} \frac{k^3}{(k+1)^{\log k}}$.
- $\sum_{k=1}^{\infty} \left(\frac{3+(-1)^k}{5}\right)^k$.

Problem 4. *Decide for which values* $p \in \mathbb{R}$ *(with explanation!) the following series convergent and which divergent:*

- $\sum_{k=1}^{\infty} \frac{1}{\log^p k}$.
- $\sum_{k=1}^{\infty} \frac{2^{kp} k!}{k^k}$.

Problem 5. *Suppose that* $a_k \geq 0$ *and that* $a_k^{1/k} \to a$ *as* $k \to \infty$. *Prove that* $\sum_{k=1}^{\infty} a_k x^k$ *converges absolutely for all* $|x| < 1/a$ *if* $a \neq 0$ *and for all* $x \in \mathbb{R}$ *if* $a = 0$.

Problem 6. *Prove that the following series converges*

- $\sum_{k=1}^{\infty} \frac{\sin kx}{k^p}$, $x \in \mathbb{R}$, $p > 0$.
- $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k}{3^k}$.